Projective equivalences of elliptic and hyperelliptic planar curves

Juan G. Alcázar, Carlos Hermoso

Universidad de Alcalá, Alcalá de Henares, Madrid (Spain)

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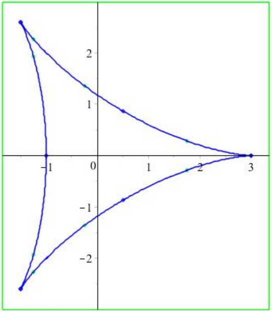
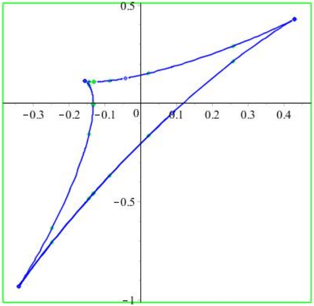
Recognizing curves up to certain transformations.



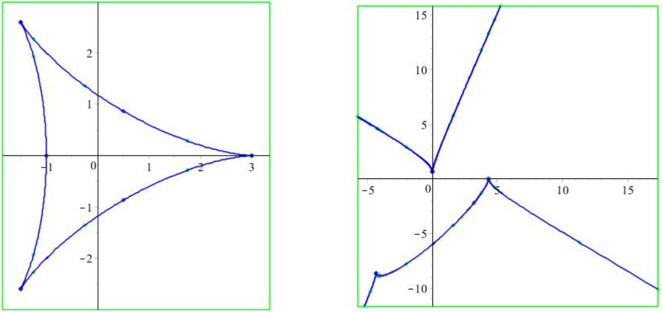


Projectivities (*projective equivalence*)

$$f(x,y) = \left(\frac{a_{11}x + a_{12}y + b_1}{a_{31}x + a_{32}y + b_3}, \frac{a_{21}x + a_{22}y + b_2}{a_{31}x + a_{32}y + b_3} \right)$$

$$\tilde{f}(\tilde{\mathbf{x}}) = \mathbf{P}\tilde{\mathbf{x}}, \ \tilde{\mathbf{x}} = [x_0 : x_1 : x_2], \mathbf{P} \in \mathbb{R}^{3\times 3}, \ \det(\mathbf{P}) \neq 0$$

Recognizing curves up to certain transformations.



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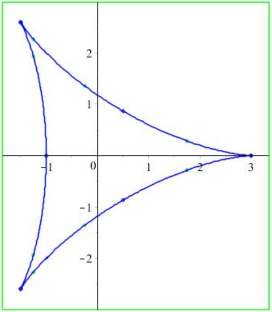
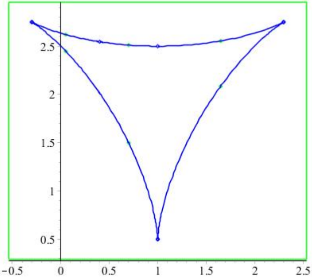


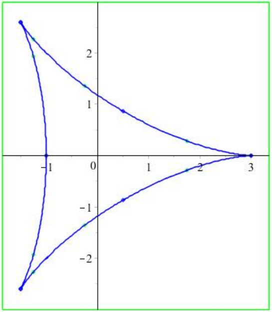


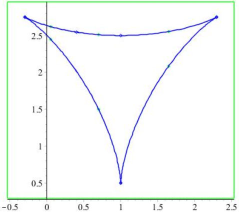
Rigid motions (*congruence*) [including symmetries of a curve]

$$f(x,y) = (\alpha x \mp \beta y + b_1, \beta x \pm \alpha y + b_2),\ \alpha^2 + \beta^2 = 1$$

$$f(\mathbf{x}) = \mathbf{Q}\mathbf{x} + \mathbf{b},\ \mathbf{Q}^T\mathbf{Q} = \mathbf{I}$$

Recognizing curves up to certain transformations.



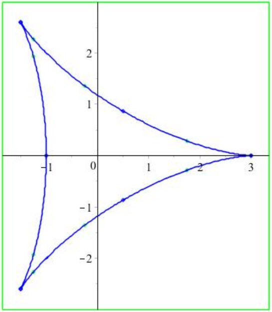


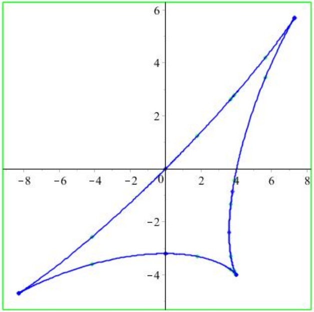
Similarities (*similaritity*)

$$f(x,y) = (\lambda(\alpha x \mp \beta y) + b_1, \lambda(\beta x \pm \alpha y) + b_2),\ \alpha^2 + \beta^2 = 1, \lambda \neq 0$$

$$f(\mathbf{x}) = \lambda \mathbf{Q}\mathbf{x} + \mathbf{b},\ \mathbf{Q}^T\mathbf{Q} = \mathbf{I},\ \lambda \neq 0$$

Recognizing curves up to certain transformations.





Affinities (*affine equivalence*)

$$f(x, y) = (a_{11}x + a_{12}y + b_1, a_{21}x + a_{22}y + b_2)$$

$$f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b},\ \mathbf{A} \in \mathbb{R}^{2\times 2},\ \det(\mathbf{A}) \neq 0$$

Projectivities.

Projectivities (*projective equivalence*)

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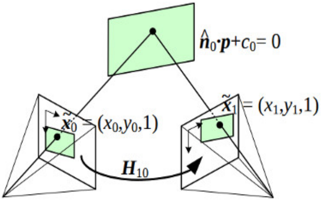

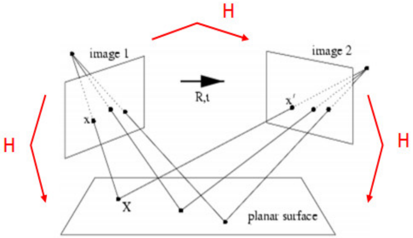

Pictures from *2D projective transformations (homographies)*, C. Gava, G. Bleser, *Computer Vision: Algorithms and Applications*, R. Szeliski

Projectivities.

Projectivities (*projective equivalence*)

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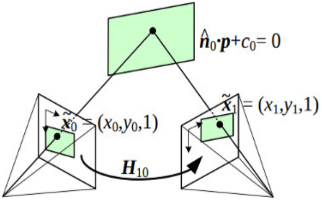
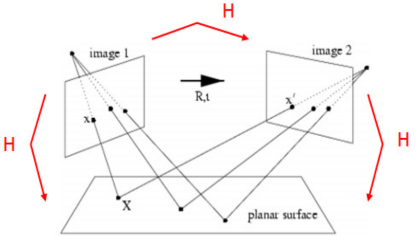




Obs.: projectivities are collineations

What curves do we want to study?

In CAGD, the preferred type of algebraic curve to study are **rational curves**, e.g.

$$\mathbf{x}(t) = \left(\frac{a(1-t^2)}{1+t^2}, \frac{2bt}{1+t^2} \right).$$

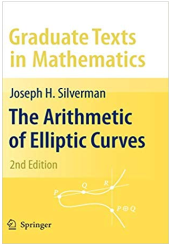
Alcázar J.G., Hermoso C., Muntingh G. (2014), *Detecting similarity of Rational Plane Curves*, Journal of Computational and Applied Mathematics vol. 269, pp. 1-13

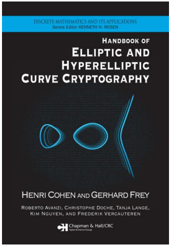
Hauer M., Jüttler B. (2018), *Projective and affine symmetries and equivalences of rational curves in arbitrary dimension*, Journal of Symbolic Computation Vol. 87, pp. 68–86.

Problem essentially solved for rational curves. Other algebraic curves?

What curves do we want to study?

Elliptic and **Hyperelliptic** curves

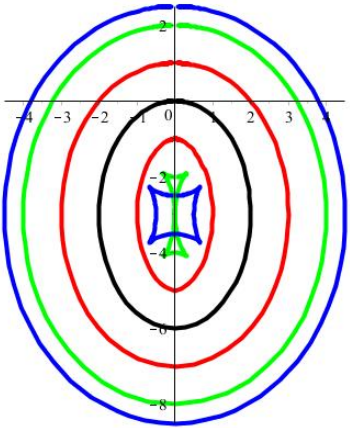




Parametrizable by square-roots of rational funcions.

Non-rational offsets of rational curves, and certain bisectors (line/rat. curve, circle/ rat. curve), are either elliptic or hyperelliptic curves.

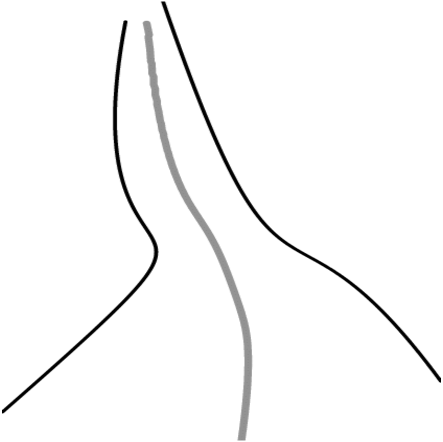
What curves do we want to study?



The ellipse is rational, but the **offsets** to the ellipse, in general are not (in general, offsets to rational curves are not rational).

What curves do we want to study?

Bisector curves of rational curves are not necessarily rational, either.



Elliptic curves

Elliptic curve: curves of *genus* 1 (genus 0 means rational) birationally equivalent to a nonsingular cubic curve (its *Weierstrass form*)

$$\mathcal{E} \stackrel{\xi}{\longmapsto} \mathcal{W},$$

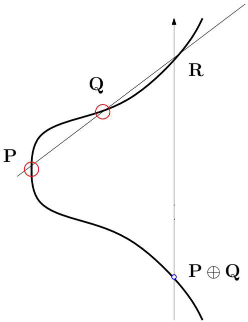
where $\mathcal{W}$ can be written as

$$y^2 = x^3 + rx + s$$

Nonsingular cubic curves have a very rich structure!

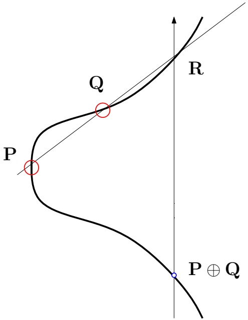
Elliptic curves

Group law in a Weierstrass curve

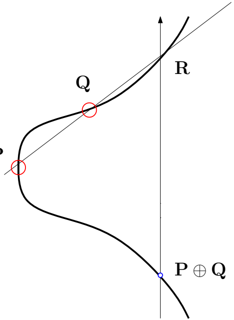


Conmmutative law (*abelian varieties*); the neutral element is $\mathcal{O} = [0 : 1 : 0]$.

Elliptic curves



- Ramification points: $P \oplus P = 2P = \mathcal{O}$.
- Flex points: $Q \oplus Q \oplus Q = 3Q = \mathcal{O}$.
- Aligned points: $P \oplus Q \oplus R = \mathcal{O}$.



Translation map by (fixed) P: $\tau_P(Q) = P \oplus Q$; if $P = (\alpha, \beta)$,

$$\tau_P(x,y) = \left(-\left(\frac{y-\beta}{x-\alpha}\right)^2 - x - \alpha, \left(\frac{y-\beta}{x-\alpha}\right)\left(-\left(\frac{y-\beta}{x-\alpha}\right)^2 - x - 2\alpha\right) + \beta \right)$$

Projective equivalences between elliptic curves

Theorem

Let $\mathcal{E}_1, \mathcal{E}_2 \subset \mathbb{R}^2$ be two elliptic curves, with Weierstrass forms $\mathcal{W}_1, \mathcal{W}_2 \in \mathbb{R}^2$, such that there exists a projectivity g mapping $\mathcal{E}_1$ to $\mathcal{E}_2$. Then there exists a birational transformation φ_g of $\mathbb{R}^2$, associated with g, mapping $\mathcal{W}_1$ onto $\mathcal{W}_2$, making the following diagram commutative:

$$\begin{array}{ccc} \mathcal{E}_1 & \xrightarrow{g} & \mathcal{E}_2 \\ \downarrow{\scriptstyle \xi_1} & & \downarrow{\scriptstyle \xi_2} \\ \mathcal{W}_1 & \xrightarrow[\varphi_g]{} & \mathcal{W}_2 \end{array} \tag{1}$$

In particular, for a generic point $(x, y) \in \mathcal{E}_1$ we have

$$\xi_2 \circ g = \varphi_g \circ \xi_1 \tag{2}$$

Assuming $\mathcal{W}_i \equiv y^2 = x^3 + r_i x + s_i$,

Theorem (A., Hermoso)

The birational transformation φ_g satisfies that

$$\varphi_g = \tau_P \circ \phi,$$

with $P \in \mathcal{W}_2$,

$$\phi = (a^2 x, a^3 y)$$

and $a \neq 0$ is a real root of $gcd(r_2 - r_1 a^4, a^6 s_1 - s_2)$.

Projective equivalences between elliptic curves

For **cubic elliptic curves** $\mathcal{E}_i$, the $\mathcal{E}_i$ and the $\mathcal{W}_i$ are projectively equivalent!

$$\begin{array}{ccc} \mathcal{E}_1 & \xrightarrow{\ g\ } & \mathcal{E}_2 \\ \big\downarrow{\scriptstyle \xi_1} & & \big\downarrow{\scriptstyle \xi_2} \\ \mathcal{W}_1 & \xrightarrow[\varphi_g]{} & \mathcal{W}_2 \end{array} \tag{3}$$

with

$$\varphi_g = \xi_2 \circ g \circ \xi_1^{-1} \tag{4}$$

So $\varphi_g = \tau_P \circ \phi$ must be a projectivity, and also τ_P (restricted to $\mathcal{W}_2$)!!

Projective equivalences between elliptic curves

Cubic elliptic curves: since projectivities are collineations, for $Q, R, S \in \mathcal{W}_2$ aligned

$$\tau_P(Q \oplus R \oplus S) = \tau_P(\mathcal{O}) = (P \oplus Q) \oplus (P \oplus R) \oplus (P \oplus S) = 3P = \mathcal{O}$$

Theorem (A., Hermoso)

The projectivities $g : \mathcal{E}_1 \to \mathcal{E}_2$, with $\mathcal{E}_i$ a cubic elliptic curve, are the mappings

$$g = \xi_2^{-1} \circ \varphi_g \circ \xi_1,$$

with $\varphi_g = \tau_P \circ \phi$, where $P = \mathcal{O}$ (i.e. τ_P is the identity) or P is a flex point of $\mathcal{W}_2$, and

$$\phi = (a^2 x, a^3 y),$$

with $a \neq 0$ a real root of $gcd(r_2 - r_1 a^4, a^6 s_1 - s_2)$.

For non-cubic elliptic curves, $P = (\alpha, \beta)$ must be included as an unknown in the computation (polynomial system solving, instead of linear system solving).

Hyperelliptic curves

Hyperlliptic curve: curves of *genus* $\kappa \geq 2$ birationally equivalent to a (singular) curve (its *Weierstrass form*)

$$\mathcal{H} \stackrel{\xi}{\longmapsto} \mathcal{W},$$

where $\mathcal{W}$ can be written as

$$y^2 = h(x),$$

with $h(x)$ square-free and of degree $2\kappa + 1$ or $2\kappa + 2$.

Projective equivalences between hyperelliptic curves

Theorem

Let $\mathcal{H}_1, \mathcal{H}_2 \subset \mathbb{R}^2$ be two hyperelliptic curves, with Weierstrass forms $\mathcal{W}_1, \mathcal{W}_2 \in \mathbb{R}^2$, such that there exists a projectivity g mapping $\mathcal{H}_1$ to $\mathcal{H}_2$. Then there exists a birational transformation φ_g of $\mathbb{R}^2$, associated with g, mapping $\mathcal{W}_1$ onto $\mathcal{W}_2$, making the following diagram commutative:

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In particular, for a generic point $(x, y) \in \mathcal{H}_1$ we have

$$\xi_2 \circ g = \varphi_g \circ \xi_1 \qquad (6)$$

Projective equivalences between hyperelliptic curves

Theorem (A., Hermoso)

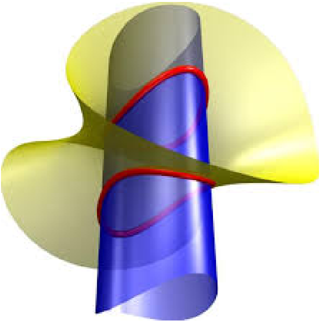
Let $\mathcal{W}_1, \mathcal{W}_2 \subset \mathbb{R}^2$ *be two hyperelliptic curves in Weierstrass form of genus* $\kappa \geq 2$. *Any real birational mapping* φ_g *between* $\mathcal{W}_1, \mathcal{W}_2$ *has the form*

$$\varphi_g(x,y) = \left(\frac{ax+b}{cx+d}, \frac{ey}{(cx+d)^{\kappa+1}} \right), \tag{7}$$

with $ad - bc \neq 0$*, and* $a, b, c, d, e \in \mathbb{R}$*,* $e \neq 0$*.*

More...

Space elliptic and hyperelliptic curves?? (e.g. intersections between a quadric and a ruled surface)



Picture from *Intersecting Quadrics: An Efficient and Exact Implementation*, S. Lazard, L. Peñaranda, S. Petitjean

More...

More on similarities, affine and projective equivalences:

Alcazar J.G., Lavicka M., Vrsek J. (2019) *Symmetries and similarities of planar algebraic curves using harmonic polynomials*, Journal of Computational and Applied Mathematics Vol. 357, pp. 302–318.

Bizzarri M., Lavicka M., Vrsek J. (2018), *Computing projective equivalences of special algebraic varieties*, ArXiv 1806.05827.

Hauer M., Jüttler B., Schicho J. (2018), *Projective and affine symmetries and equivalences of rational and polynomial surfaces*, Journal of Computational and Applied Mathematics Vol. 349, pp. 424–437.

Thanks for your attention!!