1.3 Local study of curves.
1.3.2 Curvature. Normal principal vector and osculating plane

Definition We call curvature vector of a curve $C$ with natural parametrization $\beta$ at the point $P = \beta(s)$, to the vector $\beta''(s)$. Its module $||\beta''(s)||$ will be called curvature of $C$ at the point $P$ and it defines the curvature function

$$\kappa : J \longrightarrow \mathbb{R}, \kappa(s) = ||\beta''(s)||.$$ 

Geometric interpretation. Since the tangent vector $T(s) = \beta'(s)$ has norm 1, $\kappa(s)$ measures the rate of change of the angle between $T(s)$ and the tangent vectors to $C$ at points close to $P$.

This is, if $\theta(s)$ is the angle between $\beta'(s)$ and $\beta'(s_0)$:

$$\lim_{s \to s_0} \frac{\theta(s)}{|s - s_0|} = \lim_{s \to s_0} \frac{||\beta'(s) - \beta'(s_0)||}{|s - s_0|} = ||\beta''(s)|| = \kappa(s).$$
Observations.

1. $k(s) = 0, \forall s \in J \iff C$ is a line.

2. If $\kappa(s) \neq 0$ then $\beta''(s)$ and the tangent vector $\beta'(s)$ to the curve $C$ at $P = \beta(s)$ are orthogonal, since

$$\beta'(s) \cdot \beta'(s) = 1 \Rightarrow 2\beta''(s)\beta'(s) = 0.$$ 

Definition. Let us assume that $k(s) \neq 0$ at $P = \beta(s)$. We call normal principal vector to $C$ at $P = \beta(s)$ to the unitary vector $N(s)$ in the direction of the curvature vector

$$N(s) = \frac{\beta''(s)}{||\beta''(s)||} = \frac{\beta''(s)}{\kappa(s)}.$$

We obtain the 1st Frenet-Serret Formula

$$T'(s) = \kappa(s)N(s).$$
Definition The normal line to the curve $C$ at $P = \beta(s)$ is the line going through $P$ and with direction vector, the normal principal vector $N(s)$ to $C$ at $P$. Thus, if $N(s) = (n_1, n_2, n_3)$ then the normal line $n$ to $C$ at $P = (p_1, p_2, p_3)$ has natural parametrization

$$n(\lambda) = P + \lambda N(P) = (p_1 + \lambda n_1, p_2 + \lambda n_2, p_3 + \lambda n_3).$$

Definition The osculating plane of $C$ at $P = \beta(s)$ is the plane going through $P$ and with direction vectors $T(s)$ and $N(s)$.

A point $X = (x, y, z)$ belongs to the osculating plane $\Omega$ to $C$ at $P$ if the vector $PX = (x - p_1, y - p_2, z - p_3)$ is a linear combination of $T(P) = (v_1, v_2, v_3)$ and $N(s)$, from where we obtain the implicit equation of $\Omega$,

$$\Omega \equiv \det \begin{pmatrix} x - p_1 & y - p_2 & z - p_3 \\ v_1 & v_2 & v_3 \\ n_1 & n_2 & n_3 \end{pmatrix} = 0.$$
Example

Let $C$ be the helix with natural parametrization

$$\beta(s) = \left( \frac{\sin(s)}{\sqrt{2}}, \frac{s}{\sqrt{2}}, \frac{\cos(s)}{\sqrt{2}} \right), \ s \in [0, +\infty).$$

Let us compute the equations at $P = \beta(\pi/2) = (1/\sqrt{2}, \pi/2\sqrt{2}, 0)$ of the

normal line $n(\lambda) = P + \lambda N(\pi/2) = (1/\sqrt{2} - \lambda, \pi/2\sqrt{2}, 0)$

osculating plane $\Omega \equiv \pi/4 - y/\sqrt{2} - z/\sqrt{2} = 0$. 
Definition The osculating circle of the curve $C$ at $P = \beta(s)$ is the circle contained in the osculating plane $\Omega$ to $C$ at $P$, with center in the normal line $n$ and radius $R(s) = \frac{1}{\kappa(s)}$. The center of the osculating circle is called center of curvature.

Observations

1. The osculating circle, is the circle of maximum contact with the curve. The tangent and principal normal vector at $P$ of $C$ and of the osculating circle coincide.

2. The center $Z = (c_1, c_2, c_3)$ of the osculating circle verifies $PZ = R(s)N(s)$.

3. If a point $X = (x, y, z)$ belongs to the osculating circle $||ZX|| = R(s)$. That is, the points of the osculating circle are points of intersection of the osculating plane $\Omega$ and the sphere

$$ (x - c_1)^2 + (y - c_2)^2 + (z - c_3)^2 = R(s)^2. $$
Example We continue with the helix given by its natural parametrization
\[ \beta(s) = \left( \frac{\sin(s)}{\sqrt{2}}, \frac{s}{\sqrt{2}}, \frac{\cos(s)}{\sqrt{2}} \right), \quad s \in [0, +\infty). \]
The osculating circle of \( C \) at \( P = \beta(\pi/2) = \left( \frac{1}{\sqrt{2}}, \frac{\pi}{2\sqrt{2}}, 0 \right) \) is the intersection of the sphere with center \( Z \) and radius \( R(\pi/2) = \frac{1}{\kappa(\pi/2)} = \sqrt{2} \) with the osculating plane \( \Omega \) of \( C \) at \( P \), where
\[ Z = P + R(\pi/2)N(\pi/2) = \left( \frac{-1}{\sqrt{2}}, \frac{\pi}{2\sqrt{2}}, 0 \right). \]
The sphere has equation
\[ \left( x - \frac{-1}{\sqrt{2}} \right)^2 + \left( y - \frac{\pi}{2\sqrt{2}} \right)^2 + z^2 = 2. \]
1.3.3 Binormal line and rectifying plane. Torsion

Definition Let $P = \beta(s)$ be a point of $C$ with $k(s) \neq 0$. The unitary vector $B(s) = T(s) \wedge N(s)$ is orthogonal to the osculating plane and it is called binormal vector to $C$ at $P$.

Definition The binormal line to $C$ at $P$ is the line going through $P$ and its direction vector is the binormal vector $B(s)$ to $C$ at $P$.

Therefore, if $B(s) = (b_1, b_2, b_3)$ is the binormal line $b$ to $C$ at $P = (p_1, p_2, p_3)$, its natural parametrization is

$$b(\lambda) = P + \lambda B(s) = (p_1 + \lambda b_1, p_2 + \lambda b_2, p_3 + \lambda b_3).$$

Definition The rectifying plane to $C$ at $P = \beta(s)$ is the plane going through $P$ and with direction vectors $T(s)$ and $B(s)$. 
A point \( X = (x, y, z) \) belongs to the rectifying plane \( \Delta \) of \( C \) at \( P \) if the vector 
\[
PX = (x - p_1, y - p_2, z - p_3)
\]
is orthogonal to \( N(s) = (n_1, n_2, n_3) \) or if it is a linear combination of the vectors \( T(s) \) and \( B(s) \). From these conditions, the implicit equation of \( \Delta \) is obtained
\[
(x - p_1)n_1 + (y - p_2)n_2 + (z - p_3)n_3 = 0
\]
or
\[
\det \begin{pmatrix}
  x - p_1 & y - p_2 & z - p_3 \\
  v_1 & v_2 & v_3 \\
  b_1 & b_2 & b_3
\end{pmatrix}.
\]

**Geometric interpretation** We observe that the binormal vector \( B(s) \) is orthogonal to the osculating plane \( \Omega \) to \( C \) at \( P \). Since \( B(s) \) has norm 1, the length \( \|B'(s)\| \) measures the rate of change of the angle between the osculating plane to \( C \) at \( P \) and the osculating planes to points of \( C \) close to \( P \). That is, \( B'(s) \) measures how fast the curve \( C \) goes away from the osculating plane at \( P \).
Example Let $C$ be the helix given by the natural parametrization

$$\beta(s) = \left( \frac{\cos(s)}{\sqrt{2}}, \frac{\sin(s)}{\sqrt{2}}, \frac{s}{\sqrt{2}} \right), \quad s \in [0, +\infty).$$

The binormal line at $P = \alpha(\pi)$ is (in red)

$$b(\lambda) = P + \lambda B(\pi) = [-1/\sqrt{2}, 0, \pi/\sqrt{2}] + \lambda[0, 1/\sqrt{2}, 1/\sqrt{2}],$$

the rectifying plane is (in grey)

$$\Gamma \equiv (X - P) \cdot N(\pi) = 1 + \sqrt{2}x = 0.$$
We can show that:

\[ B'(s) = T'(s) \wedge N(s) + T(s) \wedge N'(s) \]

and since \( T'(s) = \kappa(s)N(s) \) then \( T'(s) \wedge N(s) = 0 \), so

\[ B'(s) = T(s) \wedge N'(s). \]

Thus in one hand \( B'(s) \) is orthogonal to \( T(s) \) and on the other

\[ B(s) \cdot B(s) = 1 \Rightarrow B'(s) \cdot B(s) = 0, \]

that is, \( B'(s) \) is also orthogonal to \( B(s) \). We conclude that \( B'(s) \) is proportional to \( N(s) \), we obtained the 2nd Frenet-Serret Formula

\[ B'(s) = \tau(s)N(s) \text{ and } \tau(s) = B'(s) \cdot N(s) \]
Definition Given a curve $C$ such that $\kappa(s) \neq 0$, $\forall s \in J$ (equivalently $\beta''(s) \neq 0$). The number $\tau(s)$ such that $B'(s) = \tau(s)N(s)$ is called torsion of $C$ at $P$.

Summarizing, given a curve $C$ parametrized by the arc length, with natural parametrization $\beta : J \rightarrow \mathbb{R}^3$ and $\kappa(s) \neq 0$, $\forall s \in J$, we have

$$T(s) = \beta'(s), \; N(s) = \beta''(s)/\kappa(s), \; B(s) = T(s) \wedge N(s).$$

It holds that

$$\tau(s) = -\frac{[\beta'(s), \beta''(s), \beta'''(s)]}{\kappa(s)^2},$$

with $[\beta'(s), \beta''(s), \beta'''(s)]$ the mixed product of the vectors $\beta'(t)$, $\beta''(t)$ and $\beta'''(t)$.

Proposition Let $C$ be a curve such that $\kappa(s) \neq 0$, $\forall s \in J$. Then,

$$C \text{ is a plane curve } \iff \tau(s) = 0, \forall s \in J.$$
Proof $C$ is contained in a plane if and only if $B(s)$ is a constant vector $B_0 = (b_1, b_2, b_3)$, for every $s \in J$. Equivalently, $B'(s) = \mathbf{0} = (0, 0, 0)$, $\forall s \in J$ and $\tau(s) = 0$, $\forall s \in J$.

Observe that the torsion is positive or negative, whereas the curvature is always positive.

Example Let $C$ be an helix parametrized by:

$$\beta(s) = \left( a \cos \left( \frac{s}{c} \right), a \sin \left( \frac{s}{c} \right), \frac{bs}{c} \right), s \in \mathbb{R},$$

with $a^2 + b^2 = c^2 \neq 0$. Prove that $\beta$ is the natural parametrization of $C$, that $\kappa(s) \neq 0$, $\forall s \in \mathbb{R}$ and compute the torsion $\tau(s)$, $\forall s \in \mathbb{R}$.

Since $||\beta'(s)|| = 1$ then $\beta$ is the natural parametrization of $C$ and $s$ is the arc length parameter.

$$\kappa(s) = \frac{\sqrt{2}|a|}{c^2} > 0 \text{ and } \tau(s) = \frac{-b}{c^2}, \forall s \in \mathbb{R}.$$
Thus, the sign of the torsion depends of the sign of $b$.
The next are helices with values $a = 1$, $c = \sqrt{5}$ and $b = \pm 2$.

If the torsion is negative, the coordinate $z(s)$ of $\beta(s) = (x(s), y(s), z(s))$ is increasing, otherwise it is decreasing.
1.3.4 Moving frame of Frenet-Serret

To each value of the parameter \( s \in J \) we associate three unitary, orthogonal vectors \( T(s) \), \( N(s) \) and \( B(s) \).

**Definition** The orthonormal basis \( \{T(s), N(s), B(s)\} \) is called Frenet-Serret frame of \( C \) at \( P = \beta(s) \).

At each point \( P = \beta(s) \) of \( C \) we have an affine orthonormal coordinate system of \( \mathbb{R}^3 \) with origin at \( P \) and basis \( \{T(s), N(s), B(s)\} \) of \( \mathbb{R}^3 \).

The derivatives \( T'(s) = \kappa(s)N(s) \) and \( B'(s) = \tau(s)N(s) \) expressed in terms of the Frenet frame determine curvature and torsion, which provide the information about the behavior of the curve \( C \) in a neighborhood of the point \( P = \beta(s) \).

It also holds that

\[
N'(s) = B'(s) \wedge T(s) + B(s) \wedge T'(s) = -\tau(s)B(s) - \kappa(s)T(s),
\]
and we obtain the Frenet formulas

\[
\begin{align*}
  T'(s) &= \kappa(s)N(s), \\
  N'(s) &= -\tau(s)B(s) - \kappa(s)T(s), \\
  B'(s) &= \tau(s)N(s).
\end{align*}
\]

Intuitively, curvature measures the failure of a curve to be a straight line, while torsion measures the failure of a curve to be planar.

**Theorem** Given two differentiable functions \( \kappa, \tau : J \subseteq \mathbb{R} \rightarrow \mathbb{R}, \kappa(s) > 0, \forall s \in J \), there exists a unique regular parametrized curve (up to rigid movements) \( C \), with natural parametrization \( \beta : J \rightarrow \mathbb{R}^3 \) and arc length parameter \( s \) such that, \( \kappa(s) \) is the curvature and \( \tau(s) \) the torsion of \( C \) at a point \( P = \beta(s) \).
1.3.5 Frenet frame for an arbitrary parametrization

Let $C$ be a parametrized curve, with regular parametrization $\alpha : I \subseteq \mathbb{R} \rightarrow \mathbb{R}^3$, which does not have to be the natural parametrization.

We present next expressions for the elements $\{T(t), N(t), B(t)\}$ of the Frenet frame and, curvature $\kappa(t)$ and torsion $\tau(t)$, $\forall t \in I$, at that point.

\[
T(t) = \frac{\alpha'(t)}{||\alpha'(t)||}, \\
B(t) = \frac{\alpha'(t) \wedge \alpha''(t)}{||\alpha'(t) \wedge \alpha''(t)||}, \\
N(t) = B(t) \wedge T(t).
\]
In addition, the next expressions are obtained for curvature and torsion of $C$ at a point $P = \alpha(t)$:

\[
\begin{align*}
\kappa(t) & := \kappa(s(t)) = \frac{||\alpha'(t) \land \alpha''(t)||}{||\alpha'(t)||^3}, \\
\tau(t) & := \tau(s(t)) = \frac{-[\alpha'(t), \alpha''(t), \alpha'''(t)]}{||\alpha'(t) \land \alpha''(t)||^2}.
\end{align*}
\]

**Example** Compute the elements of the Frenet frame at the point $P = (0, 0, 1)$ of the curve $C$ with parametric representation

\[
\alpha(t) = (t, -t^2, 1 + t^3), \quad t \in [0, +\infty).
\]

Compute in addition curvature and torsion of the curve at a generic point of the curve.

\[
||\alpha'(t)|| = \sqrt{1 + 4t^2 + 9t^4}, \quad \text{thus } t \text{ is not the arc length parameter.}
\]