Curves

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1 Introduction

We can think of a curve as a connected one-dimensional series of points. Sometimes all points in a curve lie in a plane. These curves are called *planar curves*, in contrast to *spatial curves* that are not contained in a plane. We study geometrical concepts for planar and spatial curves. Let us start with two different analytical approaches to describing curves.

2 Representation of a curve

2.1 Parametric representation

We could consider a curve as the trace left in the space by a point that is moving. Hence the coordinates of a generic point P of the curve C are functions

$$x = x(t),\ y = y(t),\ z = z(t),$$

called *coordinate functions*, where t is assuming all the values in an interval $I \subseteq \mathbb{R}$; that is, every value of the parameter is mapped to a point $P \in C$. The map

$$\begin{array}{rcl} \vec{r}\colon I & \longrightarrow & \mathbb{R}^3, \\ \vec{r}(t) & = & (x(t), y(t), z(t)) \,, \end{array}$$

is called a *parametric representation* or *parametrization* of C. In the following examples we see how a curve can be given by different parametric representations.

Examples

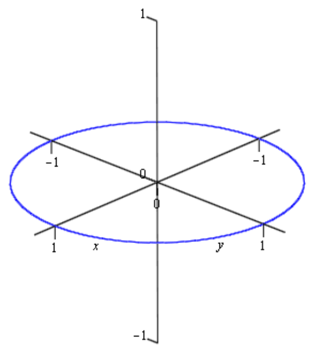
1. *Straight line.* A straight line containing a point of coordinates (a_1, a_2, a_3) with director vector (v_1, v_2, v_3) is given by the following parametric representation:
$$\vec{r}(t) = (a_1 + tv_1, a_2 + tv_2, a_3 + tv_3)\,,$$
where a_i, v_i are constants and at least one $v_i \neq 0$.

2. *Circle.* The circle is the set of all points in a plane that are at a given distance from a given point, the centre.

Let consider the circle C with radius r, center at the point of the coordinates $(a_1, a_2, 0)$ and contained in the plane $z = 0$. A parametric representation for C is given by

$$\vec{r}(t) = (a_1 + r\cos\alpha,\ a_2 + r\sin\alpha, 0)\,,\ \alpha \in [0, 2\pi),$$

where α is the angle formed by the point $P = \vec{r}(0)$, the origin of the coordinate O and the generic point $X = \vec{r}(t)$ of the circle.



Another parametric representation for C is:

$$\vec{r}(t) = (a_1 + r\cos 2\alpha,\ a_2 + r\sin 2\alpha, 0)\,,\ \alpha \in [0, \pi),$$

where we go over the circle at double speed.

Remark. As the functions in the previous parametrization of the circle are trigonometric functions the parametrization said to be *trigonometric.*

By solving the cartesian equation of the circle of radio r and center (a_1, a_2), that is, $(x(t) - a_1)^2 + (y(t) - a_2)^2 = r^2$, for x we obtain $y = a_2 + \sqrt{r^2 - (x - a_1)^2}$. Hence, we can consider the following parametrization of the circle,

$$\vec{r}(x) = \left(x,\ a_2 + \sqrt{r^2 - (x - a_1)^2}, 0\right),\ x \in [a_1 - r, a_1 + r].$$

Remark. The previous parametrization is said to be *irrational* as its coordinates functions are irrational. The parametrization is said to be *rational* (resp. *polynomial*) if it depends on rational (resp. polynomial) functions.

Less well-known is the parameterization of the circle by rational functions. For simplicity, let consider the unit circle in $\mathbb{R}^2$ of center the origin of coordinates. The line through the point of coordinates $(0,1)$ with slope m is given by $y = 1+mx$. This line intersects the unit circle in one other point P and as we vary m we strike every point on the unit circle. The coordinates (x,y) of P satisfy both $x^2+y^2 = 1$ and $y = 1+mx$ so we have

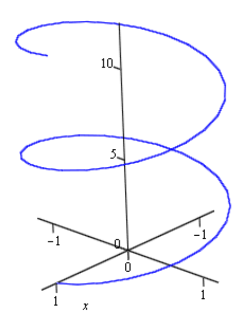
$$\begin{cases} x^2 + (1+mx)^2 = 1 \\ y = 1+mx \end{cases} \Longrightarrow \begin{cases} x^2\left(1+m^2\right) + 2mx = 0 \\ y = 1+mx \end{cases}$$
$$\Longrightarrow \begin{cases} x = \frac{-2m}{1+m^2} \text{ or } x = 0 \\ y = m(x+1) \end{cases}$$

Therefore $P = \left(\frac{-2m}{1+m^2}, \frac{1-m^2}{1+m^2}\right)$ or $P = (0,1)$ and every point on the unit circle is of the form $\left(\frac{-2m}{1+m^2}, \frac{1-m^2}{1+m^2}\right)$ and

$$\vec{r}(m) = \left(\frac{-2m}{1+m^2}, \frac{1-m^2}{1+m^2}\right),\ m \in \mathbb{R},$$

is a rational parametrization of the circle.

3. *Circular helix.* Let consider the curve contained in a circular cylinder, for example the cylinder $x^2+y^2 = r^2$ such that when x and y get again their initial values, z has increased $2\pi b$. Here's an illustration for the circular helix:



A parametric representation of the circular helix is given by

$$\vec{r}(t) = (r\cos t,\ r\sin t,\ bt)\,,\ t \in [0, 2k\pi), \text{ with } k \in \mathbb{N}.$$

The parameter t is a measure of the angle that forms the x axis with the straight line that joins the point O with the projection of the generic point $P \in C$ with the plane xy.

2.1.1 Allowable change of parameter

Definition. We say that a function $t = t(s)$, $s \in J \subseteq \mathbb{R}$, is an *allowable change of parameter* if it verifies the following conditions:

1. $t = t(s)$ is a differentiable function of class 3,
2. $t\,'(s) \neq 0$, $\forall s \in J$.

Example The function $t(\theta) = \tan\frac{\theta}{2}$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, is an allowable change of parameter for any regular parametric representation $\vec{r}(t)$, $t \in I \subseteq \mathbb{R}$, of a curve C because the function $t(\theta) = \tan\frac{\theta}{2}$ is a function of class 3 in the interval $J = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and, moreover,

$$\frac{dt}{d\theta}(\theta) = \frac{1}{2\cos^2\frac{\theta}{2}} = \tfrac{1}{2}\left(1 + \tan^2\frac{\theta}{2}\right) \neq 0,\ \forall\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

2.1.2 Regular parametric representation.

Definition. The *tangent vector* of a curve C with parametrization $\vec{r}(t) = (x(t), y(t), z(t))$, $t \in I \subseteq \mathbb{R}$, at a point $P = \vec{r}(t_0)$ is the vector

$$\vec{r}\,'(t) = (x'(t), y'(t), z'(t))\,.$$

Definition. The map $\vec{r}(t) = (x(t), y(t), z(t))$, $t \in I \subseteq \mathbb{R}$ is a *regular parametric representation* of a curve C if the following conditions hold:

1. $\operatorname{Im}\vec{r} = C$,
2. $\vec{r}$ is a differentiable application of class C^3,
3. The tangent vector to the curve at any point $P \in C$ is never zero; that is,
$$\vec{r}\,'(t) = (x'(t), y'(t), z'(t)) \neq \vec{0} \text{ for every } t \in I.$$

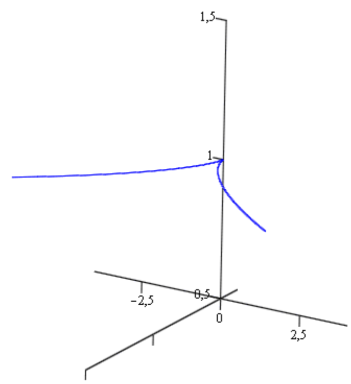
Definition. Let C be a curve with parametric representation $\vec{r}(t) = (x(t), y(t), z(t))$, $t \in I \subseteq \mathbb{R}$.

- A point $P = \vec{r}(t_0) \in C$ is said to be *singular* if $\vec{r}\,'(t_0) = \vec{0}$. If $\vec{r}\,'(t_0) \neq \vec{0}$ otherwise it is said to be *regular*.

- A point $P = \vec{r}(t_0) \in C$ is said to be a *double point* if there are $t_1, t_2 \in I$, $t_1 \neq t_2$, so that $\vec{r}(t_1) = \vec{r}(t_2) = P$.

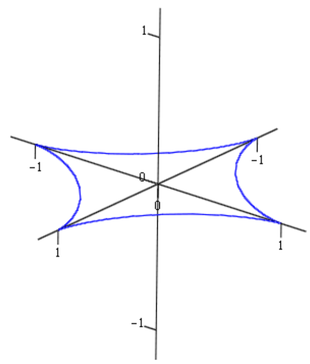
Examples

1. The polynomial parametrization $\vec{r}(t) = (t^2, t^3, 1)$ define a planar curve (contained in the plane $z = 1$). As $\vec{r}$ is a polyonimal parametrization it is differentiable of class C^∞.



   We have: $\vec{r}\,'(t) = (2t, 3t^2, 0)$. Being $\vec{r}\,'(0) = \vec{0}$ the point $P = \vec{r}(0) = (0, 0, 1)$ is a singular point of the curve. The other point of the curves are regular points because $\vec{r}\,'(t) \neq \vec{0}$ if $t \neq 0$.

2. The parametrization $\vec{r}(t) = \left(\cos^3 t,\ \sin^3 t,\ 0\right)$, $t \in [0, 2\pi)$, define a planar curve (contained in the plane $z = 0$). As $\vec{r}$ is a trigonometric parametrization it is differentiable of class C^∞.



We have: $\vec{r}\,'(t) = \big(-3\cos^2 t \sin t,\ 3\sin^2 t\cos t,\ 0\big)$. Hence $\vec{r}\,'(t) = (0,0,0) \Leftrightarrow \sin t = 0$ or $\cos t = 0$; that is, if $t = 0, \pi/2, \pi, 3\pi/2$. Thus, the points

$$\begin{array}{ll} \vec{r}(0) = (1,0,0)\,, & \vec{r}(\pi/2) = (0,1,0)\,, \\ \vec{r}(\pi) = (-1,0,0)\,, & \vec{r}(3\pi/2) = (0,-1,0)\,, \end{array}$$

are singular points.

2.2 Implicit representation

A curve C can also be considered as the intersection of two surfaces. Hence, if $F(x,y,z) = 0$ and $G(x,y,z) = 0$ are the respective equations of two surfaces then the coordinates (x,y,z) of a generic point P of the curve C must satisfy both equations: $F(x,y,z) = 0$ and $G(x,y,z) = 0$. The equations $F(x,y,z) = 0$, $G(x,y,z) = 0$ are called *cartesian or implicit equations* of the curve.

Examples

1. *Straight line.* A straight line can be consider as the intersection of two planes. For example the straight line with parametrization $\vec{r}(t) = (a_1 + tv_1,\ a_2 + tv_2,\ a_3 + tv_3)$, $t \in \mathbb{R}$ can be seen as the intersection of the planes π_1, π_2 with equations:

$$\pi_1 \equiv \frac{x - a_1}{v_1} = \frac{y - a_2}{v_2}, \text{ and } \pi_2 \equiv \frac{x - a_1}{v_1} = \frac{z - a_3}{v_3}.$$

2. *Circle.*

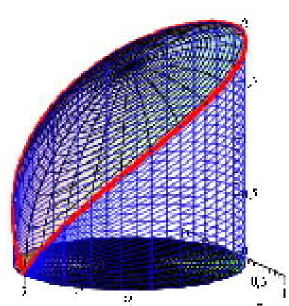
   A circle can be seen as the intersection of a sphere and a plane. For example, the unit cicle in the plane $z = 0$ ca be seen as the intersecction of the sphere of equation $x^2 + y^2 + z^2 = a^2$ and the plane $z = 0$.. It can also be seen as the intersection of the paraboloid of equation $x^2 + y^2 - z = a^2$ with the plane of equation $z = 0$ or as the intersection of the cylinder $x^2 + y^2 = a^2$ and the plane $z = 0$.

3. *Viviani's curve.* This curve is defined as the intersection of the hemisphere with a cylinder whose axis is parallel to a diameter of the sphere.

   For example, let us consider the hemisphere with center the origin of coordinates and radius 2 and the cylinder with axis the straight line $x = 0, y = 0$ and basis the circle with center $Z(1, 0, 0)$ and radius 1. Hence the Viviani's curve is the set of points satisfying

$$\begin{cases} x^2 + y^2 + z^2 = 4, \quad z \geq 0, \text{ (implicit equation of the hemisphere)}, \\ (x-1)^2 + y^2 = 1, \text{ (implicit equation of the cylinder)}. \end{cases}$$

   See the illustration bellow for a Viviani's curve:



   We are going to find a parametric representation of this curve. The circle $(x-1)^2 + y^2 = 1$ in $z = 0$ can be parameterized as follow:

$$x(t) = 1 + \cos t, \ y(t) = \sin t, \ z(t) = 0,$$

with $t \in [0, 2\pi)$. By substituting the values for x and y in the hemisphere equation we obtain:

$$
\begin{aligned}
x^2 + y^2 + z^2 = 4 \iff & (1 + \cos t)^2 + (\sin t)^2 + z^2 = 4 \\
\iff & 1 + 2\cos t + \cos^2 t + \sin^2 t + z^2 = 4 \\
\iff & z^2 = 2 - 2\cos t \\
\iff & z^2 = 2 - 2\left(\cos^2 \tfrac{t}{2} - \sin^2 \tfrac{t}{2}\right) \\
\iff & z^2 = 2\left(\cos^2 \tfrac{t}{2} + \sin^2 \tfrac{t}{2}\right) - 2\cos^2 \tfrac{t}{2} + 2\sin^2 \tfrac{t}{2} \\
\iff & z^2 = 4\sin^2 \tfrac{t}{2} \\
\iff & z = 2\sin \tfrac{t}{2} \text{ as } z \geq 0.
\end{aligned}
$$

Note that for values of $t \in [0, 2\pi)$, $\sin \frac{t}{2}$ is always positive or null. Thus, one parametrization of the Viviani's curve is:

$$\vec{r}(t) = \left(1 + \cos t,\ \sin t,\ 2\sin \tfrac{t}{2}\right), t \in [0, 2\pi).$$

3 The length of a curve

The first and simplest geometrical quantity associated with a curve is its length.

Definition. Let C be a curve with regular parametric representation $\vec{r}(t) = (x(t), y(t), z(t))$, $t \in I$. Then the *length* of C over the interval $I = (a, b)$ is given by

$$\text{length}[\vec{r}] = \int_a^b ||\vec{r}\,'(u)||du.$$

Remark. Let $\vec{\alpha}$ be a reparametrization of $\vec{r}$. Then

$$\text{length}[\vec{r}] = \text{length}[\vec{\alpha}].$$

That is the lenght of ac urve does not depend on the parametrization used to compute it.

Definition. Fix a number $a < c < b$. The *arc length function* s of a curve C with regular parametric representation $\vec{r}$: $(a, b) \to \mathbb{R}^3$ starting at c is defined by

$$s(t) = \int_c^t ||\vec{r}\,'(u)||du, \text{ where } t \in (a, b).$$

Remark. Note that

$$s\,'(t) = ||\vec{r}\,'(t)|| \neq 0, \ \forall t \in I,$$

as $\vec{r}(t)$ is a regular parametric representation. Therefore the arc length is an allowable change of parameter. As $s\,'(t) \neq 0$, $\forall t \in I$,the Inverse Function Theorem implies that $t \longmapsto s(t)$ has an inverse $s \longmapsto t(s)$ and that

$$t\,'(s) = \tfrac{1}{s\,'(t)} = \tfrac{1}{||\vec{r}\,'(t)||}.$$

Now define $\vec{\alpha}$ by $\vec{\alpha}(s) = \vec{r}(t(s))$. We have

$$\vec{\alpha}\,'(s) = \vec{r}\,'(t(s))t'(s) = \tfrac{\vec{r}\,'(t(s))}{||\vec{r}\,'(t(s))||},$$

and therefore

$$||\vec{\alpha}\,'(s)|| = \left\| \tfrac{\vec{r}\,'(t(s))}{||\vec{r}\,'(t(s))||} \right\| = \tfrac{||\vec{r}\,'(t(s))||}{||\vec{r}\,'(t(s))||} = 1.$$

Hence the unit-speed curves are said to be *parametrized by arc length* or they have a *natural parametric representation*.

Examples

1. Let C be the curve with parametric representation:

$$\vec{r}(t) = \left(\tfrac{\sqrt{2}}{2}\sin(t),\ \tfrac{\sqrt{2}}{2}t,\ \tfrac{\sqrt{2}}{2}\cos(t)\right),\ \forall\ t \in [0, 2\pi).$$

we have,

$$\vec{r}\,'(t) = \left(\tfrac{\sqrt{2}}{2}\cos(t),\ \tfrac{\sqrt{2}}{2},\ -\tfrac{\sqrt{2}}{2}\sin(t)\right),\ \forall\ t \in [0, 2\pi),$$

and

$$\begin{aligned} \|\vec{r}\,'(t)\| &= \sqrt{\left(\tfrac{\sqrt{2}}{2}\cos(t)\right)^2 + \left(\tfrac{\sqrt{2}}{2}\right)^2 + \left(-\tfrac{\sqrt{2}}{2}\sin(t)\right)^2} \\ &= \sqrt{\tfrac{1}{2}\cos^2(t) + \tfrac{1}{2} + \tfrac{1}{2}\sin^2(t)} \\ &= 1. \end{aligned}$$

Therefore t is the arc parameter and $\vec{r}$ is the natural or arc-length representation.

2. Let C be the curve with parametric representation:

$$\vec{r}(t) = (a\cos(t),\ a\sin(t),\ bt), \quad \forall t \in [0, +\infty),$$

with $a^2 + b^2 \neq 0$. We have:

$$\vec{r}\,'(t) = (-a\sin(t),\ a\cos(t),\ b), \quad \forall t \in [0, +\infty),$$

and

$$\begin{aligned} \|\vec{r}\,'(t)\| &= \sqrt{a^2\sin^2(t) + a^2\cos^2(t) + b^2} \\ &= \sqrt{a^2 + b^2}, \end{aligned}$$

thus, the arc length is given by the function:

$$s(t) = \int_0^t \|\vec{r}\,'(u)\|du = \int_0^t \sqrt{a^2 + b^2}du = \sqrt{a^2 + b^2}t,$$

that is a linear application. We have,

$$s\,'(t) = \sqrt{a^2 + b^2} \neq 0,$$

that is an allowable change of parameter. We have:

$$t(s) = \frac{s}{\sqrt{a^2+b^2}},$$

and the natural parametric representation of C is the following:

$$\vec{r}(t(s)) = \left(a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right),\ a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right),\ b\frac{s}{\sqrt{a^2+b^2}}\right),$$

where the arc parameter s takes values in the interval $[0, +\infty)$.

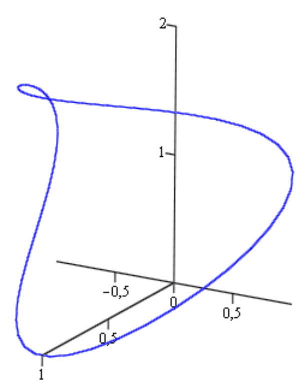
3. Let C be the curve intersection of the following surfaces:

$$\begin{aligned} S_1 &\equiv y^2 + (z-1)^2 = 1, \\ S_2 &\equiv x + y^2 = 1. \end{aligned}$$

A parametric representation of C is:

$$\vec{r}(t) = (1 - \sin^2(t), \sin(t), 1 + \cos(t)), \text{ with } t \in [0, 2\pi).$$

In the picture below you can see the plot of the curve:



We have

$$\vec{r}\,'(t) = (-2\sin(t)\cos(t), \cos(t), -\sin(t)),\ \forall\, t \in [0, 2\pi),$$

and

$$\begin{aligned} ||\vec{r}\,'(t)|| &= (-2\sin(t)\cos(t))^2 + \cos^2(t) + \sin^2(t) \\ &= 4\sin^2(t)\cos^2(t) + 1 \\ &= \sin^2(2t) + 1. \end{aligned}$$

As $||\vec{r}\,'(t)|| \neq 0$ for every t the parametrization $\vec{r}$ is regular but it is not the natural parametrization as $||\vec{r}\,'(t)|| \neq 1$, for some $t \in [0, 2\pi)$. The arc parameter function is given by the following expression:

$$s(t) = \int_0^t \sqrt{\sin^2(2u) + 1}du.$$

The inverse function of $s(t)$, which is needed to find the unit-speed parametrization, is too complicated to be of much use.

4. Let C be a curve with parametric representation:

$$\vec{r}(t) = \left((t+\pi)\cos(t),\ -(t+\pi)\sin(t),\ \tfrac{t^2}{2} + \pi t + \tfrac{\pi^2}{2}\right),\ t \in [-2\pi, 2\pi].$$

Hence

$$\vec{r}\,'(t) = (\cos(t) - (t+\pi)\sin(t),\ -\sin(t) - (t+\pi)\cos(t),\ 2t+\pi),$$

and $||\vec{r}\,'(t)||^2 = 1 + 5t^2 + 6\pi t + 2\pi^2$. Thus, $||\vec{r}\,'(t)|| = 0$ if and only if

$$\begin{aligned} 1 + 5t^2 + 6\pi t + 2\pi^2 &= 0 \Longleftrightarrow t = \tfrac{-6\pi \pm \sqrt{36\pi^2 - 20(2\pi^2+1)}}{10} \notin \mathbb{R} \\ \text{as } 36\pi^2 - 20(2\pi^2 + 1) &= -4\pi^2 - 20 < 0. \end{aligned}$$

Hence the parametrization is regular but it is not the natural parametization as $||\vec{r}\,'(t)|| \neq 1$ for some $t \in [-2\pi, 2\pi]$. The arc parameter function is given by the following expression:

$$s(t) = \int_{-2\pi}^t ||\vec{r}\,'(u)||du = \int_{-2\pi}^t \sqrt{1 + 5u^2 + 6\pi u + 2\pi^2}du.$$

The inverse function of $s(t)$, which is needed to find the unit-speed parametrization, is an elliptic function that is too complicated to be of much use.

4 Local study of unit-speed curves

We first define the curvature and torsion of a *unit-speed* curve in $\mathbb{R}^3$, as the defienition are more straightforward in this case. The case of arbitrary-speed curves in $\mathbb{R}^3$ will be considered in the next section.

Therefore during this section $\vec{r}\colon I \to \mathbb{R}^3$ will be a natural (or unit-speed) parametrization of a curve $C \subset \mathbb{R}^3$.

4.1 Curvature of unit-speed curves

Definition. The function $\kappa\colon I \to \mathbb{R}$ defined as

$$\kappa(s) = ||\vec{r}\,''(s)||,$$

is called the *curvature* of $\vec{r}$.

Intuitively, curvature measures the failure of a curve to be a straight line. In fact, a straight line is characterized by the fact that its curvature vanishes at every value of the parameter s.

Geometrical interpretation of the curvature. The curvature measures the variation of the angle between the respective tangent lines of neighboring points of the curve. Let $\theta(s)$ the angle between the tangent line to C at the point $\vec{r}(s_0)$ and the the tangent line to C at the point $\vec{r}(s)$. We have:

$$\lim_{s \to s_0} \frac{\theta(s)}{|s-s_0|} = \lim_{s \to s_0} \frac{\theta(s)}{|s-s_0|} \frac{\sin\frac{\theta(s)}{2}}{\frac{\theta(s)}{2}} = \lim_{s \to s_0} \frac{2\sin\frac{\theta(s)}{2}}{|s-s_0|},$$

and taking into account the equality

$$||\vec{r}\,'(s) - \vec{r}\,'(s_0)|| = 2\sin\tfrac{\theta(s)}{2},$$

we have:

$$\lim_{s \to s_0} \frac{2\sin\frac{\theta(s)}{2}}{|s-s_0|} = \lim_{s \to s_0} \frac{||\vec{r}\,'(s) - \vec{r}\,'(s_0)||}{|s-s_0|} = ||\vec{r}\,''(s_0)||.$$

Definition. The vector field $\vec{t}(s) = \vec{r}\,'(s)$ is called the *unit tangent vector field* of the curve C with natural parametrization $\vec{r}$. Note that the unit tangent vector field satisfies $||\vec{t}(s)|| = 1$

Remark. Let $\vec{t}(s)$ be the unit tangent vector field of a curve C, then $0 = \vec{t}(s) \cdot \vec{t}\ '(s)$ as differentiating the equation $\vec{t}(s) \cdot \vec{t}(s) = 1$ we obtain $0 = (\vec{t}(s) \cdot \vec{t}(s))' = 2\vec{t}\,'(s) \cdot \vec{t}(s)$.

We now restrict ourselves to unit-speed curves whose curvature is strictly positive (that is, $\kappa(s) > 0$). Hence, the vector field $\vec{t}\,'(s) = \vec{r}\,''(s)$ never vanishes. Let us define the *principal normal vector field* and the *binormal vector field*.

Definition. The *principal normal vector field* is the unitary vector field $\vec{n}(s)$ is the unitary vector field in the direcction of the curvature vector field; that is,

$$\vec{n}(s) = \frac{\vec{t}\,'(s)}{\kappa(s)}. \tag{1}$$

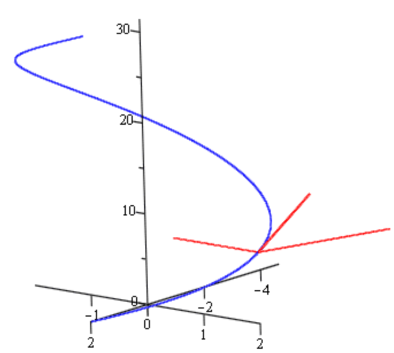
As $\vec{t}\,'(s) = \vec{r}\,''(s)$ and $\kappa(s) = ||\vec{r}\,''(s)||$. The *binormal vector field* $\vec{b}(s)$ is unitary vector field orthogonal to the tangent vector and to the principal normal vector; thas is,

$$\vec{b}(s) = \vec{t}(s) \wedge \vec{n}(s). \tag{2}$$

4.2 Frenet frame field

Definition. The triple $(\vec{t}(s), \vec{n}(s), \vec{b}(s))$ is called the *Frenet frame field* of the natural parametrization $\vec{r}(s)$ of a curve $C \in \mathbb{R}^3$. The Frenet frame field gives us an orthogonal *moving frame* along the curve.

The Frenet frame at a given point of a space curve is illustrated in the figure bellow



4.3 Torsion of unit-speed curves

In order to measure the separation of the curve from the plane containing the point $P = \vec{r}(s)$ and with characteristic vector $\vec{b}(s)$ we study the variation of the binormal vector field.

By differentiating the binormal vector field $\vec{b}(s) = \vec{t}(s) \wedge \vec{n}(s)$ we have:

$$\vec{b}'(s) = \vec{t}\,'(s) \wedge \vec{n}(s) + \vec{t}(s) \wedge \vec{n}'(s).$$

Let us compute $\vec{n}'(s)$. By differentiating the identity $\vec{n}(s) \cdot \vec{n}(s) = 1$, we obtain $\vec{n}(s) \cdot \vec{n}'(s) = 0$ and we can conclude that $\vec{n}'(s)$ is orthogonal to $\vec{n}(s)$. Therefore $\vec{n}'(s)$ is a linear combination of $\vec{t}(s)$ and $\vec{b}(s)$; that is, there exist functions $\mu(s)$, $\tau(s)$ such that

$$\vec{n}'(s) = \mu(s)\vec{t}(s) + \tau(s)\vec{b}(s). \tag{3}$$

By differentiating the identity $\vec{n}(s) \cdot \vec{t}(s) = 0$ we have:

$$\begin{aligned} 0 &= \big(\vec{n}(s) \cdot \vec{t}(s)\big)' \\ &= \vec{n}'(s) \cdot \vec{t}(s) + \vec{n}(s) \cdot \vec{t}\,'(s) \\ &= \Big(\mu(s)\vec{t}(s) + \tau(s)\vec{b}(s)\Big) \cdot \vec{t}(s) + \vec{n}(s) \cdot \kappa(s)\vec{n}(s) \\ &= \mu(s) + \kappa(s). \end{aligned}$$

Therefore $\mu(s) = -\kappa(s)$ and hence $\vec{n}'(s) = -\kappa(s)\vec{t}(s) + \tau(s)\vec{b}(s)$. By substituting $\vec{n}'(s)$ and $\vec{t}\,'(s) = \kappa(s)\vec{n}(s)$ into the expresion of $\vec{b}'(s)$ we have:

$$\begin{aligned} \vec{b}'(s) &= \vec{t}\,'(s) \wedge \vec{n}(s) + \vec{t}(s) \wedge \vec{n}'(s) \\ &= \kappa(s)\vec{n}(s) \wedge \vec{n}(s) + \vec{t}(s) \wedge \Big(\mu(s)\vec{t}(s) + \tau(s)\vec{b}(s)\Big) \\ &= \tau(s)\vec{t}(s) \wedge \vec{b}(s) \\ &= -\tau(s)\vec{n}(s). \end{aligned}$$

Finally, taking into account $\vec{n}(s) \cdot \vec{n}(s) = 1$ we obtain

$$\vec{b}'(s) \cdot \vec{n}(s) = -\tau(s).$$

Definition. The function defined as follows

$$\tau(s) = -\vec{b}'(s) \cdot \vec{n}(s).$$

is called *torsion or second curvature* of the curve $\vec{r}(s)$ the function $\tau(s)$ and it measures the variation of the binormal vector field.

4.4 Frenet-Serret formulas

Let $\vec{r}\colon I \to \mathbb{R}^3$ be a natural (or unit-speed) parametrization of a curve $C \subset \mathbb{R}^3$ with curvature $\kappa(s)$ and torsion $\tau(s)$. The vectors of the Frenet frame $\left(\vec{t}(s), \vec{n}(s), \vec{b}(s)\right)$ at any point $P = \vec{r}(s)$ of the curve, satisfy the following equations:

$$\begin{cases} \vec{t}\,'(s) = & & \kappa(s)\vec{n}(s), & \\ \vec{n}'(s) = & -\kappa(s)\vec{t}(s) & & +\tau(s)\vec{b}(s), \\ \vec{b}'(s) = & & -\tau(s)\vec{n}(s), & \end{cases}$$

called the *Frenet-Serret formulas.* The Frenet-Serret formulas are a system of ordinary differential equations for the vectors $\vec{t}(s)$, $\vec{n}(s)$ and $\vec{b}(s)$. Taking into account the theorem of existence and uniqueness of solutions of initial value problem for a system of ordinary differential equations we have the following result:

Theorem. Given two functions $\kappa, \tau\colon I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ of class C^1 with $\kappa(s) > 0$, $\forall s \in I$, there exists a unique curve $\vec{r}(s)$ up to transformations by the Euclidean group, such that $\kappa(s)$ is the curvature function of $\vec{r}(s)$ and $\tau(s)$ is its torsion function of $\vec{r}(s)$.

The above result tells us that the curvature and torsion functions; that is, $\kappa(s)$ and $\tau(s)$, determine the curve up to position at the space. For this reason, the equations $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are called *intrinsic equations of the curve.*

4.4.1 Examples

1. Let C be a curve with parametric representation $\vec{r}(s)$, $s \in I$. Let check by using the Frenet-Serret formulas that the following conditions are equivalent:

   (a) The curve $\vec{r}(s)$ is a plane curve;

   (b) $\tau(s) = 0$ for every $s \in I$.

   Note that when both conditions hold, the binormal vector field $\vec{b}(s)$ is perpendicular tio the plane containing the curve $\vec{r}(s)$.

   The condition that a curve $\vec{r}(s)$ lie in a plane Π can be expressed analytically as

   $$(\vec{r}(s) - P) \cdot \vec{q} = 0$$

where P is a point in the plane and $\vec{q}$ is an unitary vector orthogonal to Π. By differentiating the above equation we have:

$$\vec{r}\,'(s) \cdot \vec{q} = 0 \text{ and } \vec{r}\,''(s) \cdot \vec{q} = 0.$$

Thus both $\vec{t}(s)$ and $\vec{n}(s)$ are orthogonal to $\vec{q}$. Since $\vec{b}(s)$ is also orthogonal to $\vec{t}(s)$ and $\vec{n}(s)$, it follows that

$$\vec{b}(s) = \vec{q}.$$

Therefore $\vec{b}'(s) = 0$ and from the third Frenet-Serret formula we deduce $\tau(s) = 0$.

Conversely, suppose $\tau(s) = 0$, for every $s \in I$. Then the third Frenet-Serret formula $\vec{b}'(s) = -\tau(s)\vec{n}(s)$ implies $\vec{b}'(s) = 0$ for every $s \in I$, therefore $\vec{b}(s) = \vec{b}$ is a constant vector field. Let consider $s_0 \in I$ and the function

$$f(s) = (\vec{r}(s) - \vec{r}(s_0)) \cdot \vec{b}.$$

As $f(s_0) = 0$ and $f'(s) = \vec{r}'(s) \cdot \vec{b} = 0$, we have $f \equiv 0$; that is, $(\vec{r}(s) - \vec{r}(s_0)) \cdot \vec{b} = 0$ and therefore the curve is contained in the plane orthogonal to $\vec{b}$ containing the point $\vec{r}(s_0)$.

2. Let C be a *circular helix* with parametrization

$$\vec{r}(t) = (a\cos(t),\ a\sin(t),\ bt), \quad t \in [0, +\infty),$$

where the radio $a > 0$ and b is the *incline* of the helix.

As

$$\vec{r}\,'(t) = (-a\sin(t),\ a\cos(t),\ b),$$

then $||\vec{r}\,'(t)|| = \sqrt{a^2 + b^2}$. The helix is one of the few curves for which a natural parametrization is easy to find. A natural parametrization of the circular helix is

$$\vec{\alpha}(s) = \left(a\cos\tfrac{s}{\sqrt{a^2+b^2}},\ a\sin\tfrac{s}{\sqrt{a^2+b^2}},\ b\tfrac{s}{\sqrt{a^2+b^2}}\right), \quad t \in [0, +\infty).$$

Let us compute the tangent, normal and binormal vector fields and the curvature and torsion functions of the helix. We have:

$$\begin{aligned}\vec{t}(s) &= \vec{\alpha}'(s) = \left(-\tfrac{a}{\sqrt{a^2+b^2}}\sin\tfrac{s}{\sqrt{a^2+b^2}},\ \tfrac{a}{\sqrt{a^2+b^2}}\cos\tfrac{s}{\sqrt{a^2+b^2}},\ \tfrac{b}{\sqrt{a^2+b^2}}\right)\\ &= \tfrac{1}{\sqrt{a^2+b^2}}\left(-a\sin\tfrac{s}{\sqrt{a^2+b^2}},\ a\cos\tfrac{s}{\sqrt{a^2+b^2}},\ b\right),\end{aligned}$$

and

$$\vec{\alpha}''(s) = \tfrac{1}{a^2+b^2}\left(-a\cos\tfrac{s}{\sqrt{a^2+b^2}},\ -a\sin\tfrac{s}{\sqrt{a^2+b^2}},\ 0\right).$$

As $a>0$, we have

$$\kappa\,(s) = ||\vec{\alpha}''\,(s)\,|| = \tfrac{a}{a^2+b^2},$$

and

$$\begin{aligned}\vec{n}\,(s) &= \frac{\vec{\alpha}''\,(s)}{||\vec{\alpha}''\,(s)\,||}\\ &= \left(-\cos\tfrac{s}{\sqrt{a^2+b^2}},\ -\sin\tfrac{s}{\sqrt{a^2+b^2}},\ 0\right).\end{aligned}$$

By the formula $\vec{b}\,(s) = \vec{t}\,(s)\wedge\vec{n}\,(s)$, we obtain

$$\begin{aligned}\vec{b}\,(s) &= \begin{vmatrix} i & j & k\\ -\tfrac{a}{\sqrt{a^2+b^2}}\sin\tfrac{s}{\sqrt{a^2+b^2}} & \tfrac{a}{\sqrt{a^2+b^2}}\cos\tfrac{s}{\sqrt{a^2+b^2}} & \tfrac{b}{\sqrt{a^2+b^2}}\\ -\cos\tfrac{s}{\sqrt{a^2+b^2}} & -\sin\tfrac{s}{\sqrt{a^2+b^2}} & 0\end{vmatrix}\\ &= \tfrac{1}{\sqrt{a^2+b^2}}\begin{vmatrix} i & j & k\\ -a\sin\tfrac{s}{\sqrt{a^2+b^2}} & a\cos\tfrac{s}{\sqrt{a^2+b^2}} & b\\ -\cos\tfrac{s}{\sqrt{a^2+b^2}} & -\sin\tfrac{s}{\sqrt{a^2+b^2}} & 0\end{vmatrix}\\ &= \tfrac{1}{\sqrt{a^2+b^2}}\left(b\sin\tfrac{s}{\sqrt{a^2+b^2}},\ -b\cos\tfrac{s}{\sqrt{a^2+b^2}},\ a\right).\end{aligned}$$

Finally, as

$$\vec{b}\,'\,(s) = \tfrac{b}{a^2+b^2}\left(\cos\tfrac{s}{\sqrt{a^2+b^2}},\ \sin\tfrac{s}{\sqrt{a^2+b^2}},\ 0\right)$$

by comparing the expression for $\vec{b}\,'\,(s)$ with the expression for $\vec{n}\,(s)$ and by using the third Frenet-Serret formula: $\vec{b}'(s) = -\tau(s)\vec{n}(s)$ we deduce:

$$\tau(s) = \tfrac{b}{a^2+b^2}.$$

Remark. Both the curvature function and the torsion function of a helix are constant.

4.4.2 Computaion of the torsion function.

The torsion function of a curve with natural parametrization $\vec{r}(s)$ can be computed as follows:

$$\tau(s) = \frac{[\vec{r}\,'(s), \vec{r}\,''(s), \vec{r}\,'''(s)]}{||\vec{r}\,''(s)||^2}.$$

From the third Frenet-Serret formula we have: $\tau(s) = -\vec{b}'(s) \cdot \vec{n}(s)$.

By substituting

$$\begin{aligned} \vec{b}'(s) &= \big(\vec{t}(s) \wedge \vec{n}(s)\big)' \\ &= \vec{t}\,'(s) \wedge \vec{n}(s) + \vec{t}(s) \wedge \vec{n}'(s) \end{aligned}$$

into the above equations for the torsion and taking into account the expressions for $\vec{t}(s)$ and $\vec{n}(s)$ we obtain

$$\begin{aligned} \tau(s) &= -\vec{b}'(s) \cdot \vec{n}(s) \\ &= -\big(\vec{t}\,'(s) \wedge \vec{n}(s)\big) \cdot \vec{n}(s) - \big(\vec{t}(s) \wedge \vec{n}'(s)\big) \cdot \vec{n}(s) \\ &= -\big(\vec{t}(s) \wedge \vec{n}'(s)\big) \cdot \vec{n}(s) \\ &= -\left(\vec{r}\,'(s) \wedge \left(\tfrac{\vec{r}\,''(s)}{\kappa(s)}\right)'\right) \cdot \tfrac{\vec{r}\,''(s)}{\kappa(s)} \\ &= -\tfrac{1}{\kappa(s)^2} \left(\vec{r}\,'(s) \wedge \vec{r}\,'''(s)\right) \cdot \vec{r}\,''(s) \\ &= \tfrac{[\vec{r}\,'(s), \vec{r}\,''(s), \vec{r}\,'''(s)]}{\|\vec{r}\,''(s)\|^2}, \end{aligned}$$

where $[\vec{r}\,'(s), \vec{r}\,''(s), \vec{r}\,'''(s)]$ denotes the mixed product of the vectors fields $\vec{r}\,'(s)$, $\vec{r}\,''(s)$, $\vec{r}\,'''(s)$.

5 Local study of arbitrary-speed curves

For efficient computations of the curvature and torsion functions of an arbitrary-speed curve, we need formulas that avoid finding a natural parametrization explicitly.

Theorem. Let C be a curve with arbitrary regular parametric representation $\vec{r}\colon I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^3$ and nonzero curvature. Then

$$\begin{aligned}
\vec{t}(t) &= \tfrac{\vec{r}\ '(t)}{\|\vec{r}\ '(t)\|}, \\
\vec{b}(t) &= \tfrac{\vec{r}\ '(t)\wedge\vec{r}\ ''(t)}{\|\vec{r}\ '(t)\wedge\vec{r}\ ''(t)\|}, \\
\vec{n}(t) &= \vec{b}(t)\wedge\vec{t}(t), \\
\kappa(t) &= \tfrac{\|\vec{r}\ '(t)\wedge\vec{r}\ ''(t)\|}{\|\vec{r}\ '(t)\|^3}, \\
\tau(t) &= \tfrac{[\vec{r}\ '(t),\vec{r}\ ''(t),\vec{r}\ '''(t)]}{\|\vec{r}\ '(t)\wedge\vec{r}\ ''(t)\|^2}.
\end{aligned}$$

Proof. Let $\vec{\alpha}$ be the natural parametrization of C. We have: $\vec{r}(t) = \vec{\alpha}(s(t))$ and therefore

$$\begin{aligned}
\vec{r}\,'(t) &= \vec{\alpha}'(s(t))s'(t) \\
&= \vec{\alpha}'(s(t))||\vec{r}\,'(t)|| \\
&= \vec{t}(t)||\vec{r}\,'(t)||, \\
\vec{r}\,''(t) &= \vec{\alpha}''(s(t))s'(t)^2 + \vec{\alpha}'(s(t))s''(t), \\
\vec{r}\,'(t)\wedge\vec{r}\,''(t) &= \vec{\alpha}'(s(t))s'(t)\wedge\big(\vec{\alpha}''(s(t))s'(t)^2 + \vec{\alpha}'(s(t))s''(t)\big) \\
&= s'(t)^3\vec{\alpha}'(s(t))\wedge\vec{\alpha}''(s(t)) \\
&= s'(t)^3\kappa(s(t))\vec{t}(t)\wedge\vec{n}(t) \\
&= ||\vec{r}\,'(t)||^3\kappa(s(t))\vec{b}(t), \\
||\vec{r}\,'(t)\wedge\vec{r}\,''(t)|| &= ||\vec{r}\,'(t)||^3\kappa(s(t)),
\end{aligned}$$

therefore we obtain the expressions for $\vec{t}(t)$, $\vec{b}(t)$ and $\kappa(s(t))$. Moreover, taking

$$\vec{r}\,'''(t) = \vec{\alpha}'''(s(t))s'(t)^3 + 3\vec{\alpha}''(s(t))s'(t)s''(t) + \vec{\alpha}'(s(t))s'''(t),$$

and the expression of the vector $\vec{r}\,'(t)\wedge\vec{r}\,''(t)$ into account, we have:

$$\begin{aligned}
& (\vec{r}\,'(t)\wedge\vec{r}\,''(t))\cdot\vec{r}\,'''(t) \\
= \;& \left(s'(t)^3\vec{\alpha}'(s(t))\wedge\vec{\alpha}''(s(t))\right)\cdot\left(\vec{\alpha}'''(s(t))s'(t)^3+3\vec{\alpha}''(s(t))s'(t)s''(t)+\vec{\alpha}'(s(t))s'''(t)\right) \\
= \;& s'(t)^6\left(\vec{\alpha}'(s(t))\wedge\vec{\alpha}''(s(t))\right)\cdot\vec{\alpha}'''(s(t)) \\
= \;& ||\vec{r}\,'(t)||^6\left[\vec{\alpha}'(s(t)),\vec{\alpha}''(s(t)),\vec{\alpha}'''(s(t))\right] \\
= \;& ||\vec{r}\,'(t)||^6\tau(s(t))\kappa(s(t))^2 \\
= \;& ||\vec{r}\,'(t)||^6\tau(s(t))\tfrac{||\vec{r}\,'(t)\wedge\vec{r}\,''(t)||^2}{||\vec{r}\,'(t)||^6} \\
= \;& \tau(s(t))||\vec{r}\,'(t)\wedge\vec{r}\,''(t)||^2,
\end{aligned}$$

and therefore

$$\tau(t)=\tfrac{[\vec{r}\,'(t),\vec{r}\,''(t),\vec{r}\,'''(t)]}{||\vec{r}\,'(t)\wedge\vec{r}\,''(t)||^2}.$$

5.0.3 Examples

1. Let C be a curve with parametric representation:

   $$\vec{r}(t)=(t,\ -t^2,\ 1+t^3),\quad t\in[0,+\infty).$$

   Let us find the elements of the Frenet frame, the curvature and the torsion at a generic point of the curve and at the point P of coordinates $(0,0,1)$. We have:

   $$\vec{r}\,'(t)=(1,\ -2t,\ 3t^2),\quad t\in[0,+\infty).$$

   As $||\vec{r}\,'(t)||=\sqrt{1+4t^2+9t^4}\neq 1$, the parameter t is not the arc parameter. Taking the derivative of the parametric representation we obatin

   $$\begin{aligned}
   \vec{r}\,''(t) &= (0,\ -2,\ 6t), \\
   \vec{r}\,'''(t) &= (0,\ 0,\ 6) \\
   \vec{r}\,'(t)\wedge\vec{r}\,''(t) &= (-6t^2,\ -6t,\ -2), \\
   ||\vec{r}\,'(t)\wedge\vec{r}\,''(t)|| &= \sqrt{36t^4+36t^2+4} \\
   &= 2\sqrt{9t^4+9t^2+1}.
   \end{aligned}$$

Thus,

$$\begin{array}{rcl}
\vec{t}(t) & = & \frac{\vec{r}\,'(t)}{||\vec{r}\,'(t)||} = \frac{1}{\sqrt{1+4t^2+9t^4}}(1,\ -2t,\ 3t^2), \\
\vec{b}(t) & = & \frac{\vec{r}\,'(t)\wedge\vec{r}\,''(t)}{||\vec{r}\,'(t)\wedge\vec{r}\,''(t)||} = \frac{1}{2\sqrt{1+9t^2+9t^4}}(-6t^2,\ -6t,\ -2), \\
\vec{n}(t) & = & \vec{b}(t)\wedge\vec{t}(t) = \frac{1}{2\sqrt{1+9t^2+9t^4}}\frac{1}{\sqrt{1+4t^2+9t^4}}\begin{vmatrix} i & j & k \\ 1 & -2t & 3t^2 \\ -6t^2 & -6t & -2 \end{vmatrix} \\
 & = & \frac{1}{\sqrt{1+9t^2+9t^4}}\frac{1}{\sqrt{1+4t^2+9t^4}}(2t+9t^3,\ 1-9t^4,\ -3t-6t^3).
\end{array}$$

The curvature and the torsion are given by

$$\begin{array}{rcl}
\kappa(t) & = & \frac{||\vec{r}\,'(t)\wedge\vec{r}\,''(t)||}{||\vec{r}\,'(t)||^3} = \frac{2\sqrt{9t^4+9t^2+1}}{(1+4t^2+9t^4)^{3/2}}, \\
\tau(t) & = & \frac{[\vec{r}\,'(t),\vec{r}\,''(t),\vec{r}\,'''(t)]}{||\vec{r}\,'(t)\wedge\vec{r}\,''(t)||^2} = \frac{1}{36t^4+36t^2+4}\begin{vmatrix} 1 & -2t & 3t^2 \\ 0 & -2 & 6t \\ 0 & 0 & 6 \end{vmatrix} \\
 & = & \frac{-12}{36t^4+36t^2+4} = \frac{-3}{9t^4+9t^2+1}.
\end{array}$$

As $P = \vec{r}(0)$, the tangent vector, the binormal vector and the normal vector at P are:

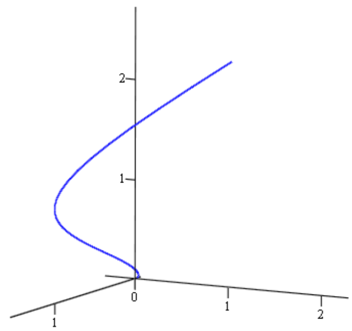
$$\begin{array}{rcl}
\vec{t}(0) & = & (1,0,0)\,, \\
\vec{b}(0) & = & (0,0,-1)\,, \\
\vec{n}(0) & = & \vec{b}(0)\wedge\vec{t}(0) = (0,-1,0)\,,
\end{array}$$

and $\kappa(0) = 2$, $\tau(0) = -3$.

2. Let C be a curve with parametric representation:

$$\vec{r}(t) = (e^t\cos(t),\ e^t\sin(t),\ e^t), \quad t\in[0,+\infty).$$

Let us find the elements of the Frenet frame, the curvature and the torsion at a generic point of the curve. In the figure bellow we have the plot of the curve:



By differentiating $\vec{r}(t)$ we obtain:

$$\begin{aligned}
\vec{r}\,'(t) &= \left(e^t\left(\cos(t)-\sin(t)\right),\ e^t\left(\sin(t)+\cos(t)\right),\ e^t\right),\\
\vec{r}\,''(t) &= \left(-2e^t\sin(t),\ 2e^t\cos(t),\ e^t\right),\\
\vec{r}\,'''(t) &= \left(-2e^t\left(\sin(t)+\cos(t)\right),\ 2e^t\left(\cos(t)-\sin(t)\right),\ e^t\right),\\
\vec{r}\,'(t)\wedge\vec{r}\,''(t) &= e^{2t}\left(\sin(t)-\cos(t),\ -\cos(t)-\sin(t),\ 2\right),\\
[\vec{r}\,'(t),\vec{r}\,''(t),\vec{r}\,'''(t)] &= 2e^{3t}.
\end{aligned}$$

Thus,

$$\kappa(t)=\tfrac{\|\vec{r}\,'(t)\wedge\vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}=\frac{e^{2t}\sqrt{6}}{e^{3t}(3)^{3/2}}=\frac{\sqrt{2}}{3e^t},$$

and

$$\tau(t)=\tfrac{[\vec{r}\,'(t),\vec{r}\,''(t),\vec{r}\,'''(t)]}{\|\vec{r}\,'(t)\wedge\vec{r}\,''(t)\|^2}=\frac{2e^{3t}}{6e^{4t}}=\frac{1}{3e^t}.$$

From

$$s=\int_0^t||\vec{r}\,'(u)||du=\int_0^t e^u\sqrt{3}du=\sqrt{3}\left(e^t-1\right),$$

we obtain

$$e^t=\tfrac{s}{\sqrt{3}}+1,$$

and therefore the intrinsic equations are

$$\kappa(s)=\tfrac{\sqrt{6}}{3\left(s+\sqrt{3}\right)},\quad \tau(s)=\tfrac{\sqrt{3}}{3\left(s+\sqrt{3}\right)}.$$

6 Lines and planes determined by the Frenet frame field

Review. The parametric equation of a line containing a point P of coordinates (a, b, c) and with director vector $\vec{v}$ of coordinates (v_1, v_2, v_3) is the following:

$$\vec{r}(\lambda) = \overrightarrow{OP} + \lambda\vec{v} \Longleftrightarrow \begin{cases} x(\lambda) = a + \lambda v_1 \\ y(\lambda) = b + \lambda v_2, \\ z(\lambda) = c + \lambda v_3, \end{cases} \quad \text{where } \lambda \in \mathbb{R}.$$

The cartesian equation of a plane Π containing a point P of coordinates (a, b, c) and with characteristic vector $\vec{v}$ of coordinates (v_1, v_2, v_3) is the following:

$$\begin{aligned} X \quad \in \quad & \Pi \Longleftrightarrow \overrightarrow{PX} \cdot \vec{v} = 0 \\ \Longleftrightarrow \quad & (x-a, y-b, z-c) \cdot (v_1, v_2, v_3) = 0 \\ \Longleftrightarrow \quad & (x-a)v_1 + (y-b)v_2 + (z-c)v_3 = 0. \end{aligned}$$

The parametric equation of a plane Π containing a point P of coordinates (a, b, c) and generated by the vector $\vec{u} = (u_1, u_2, u_3)$ and $\vec{w} = (w_1, w_2, w_3)$ is the following:

$$\vec{r}(s,t) = \overrightarrow{OP} + s\vec{u} + t\vec{w} \Longleftrightarrow \begin{cases} x(\lambda) = a + su_1 + tw_1, \\ y(\lambda) = b + su_2 + tw_2, \\ z(\lambda) = c + su_3 + tw_3, \end{cases} \quad \text{where } s, t \in \mathbb{R}.$$

Equivalently, a point X belongs to the plane Π if and only if $\overrightarrow{PX}$ is a linear combination of the vectors $\vec{u}$ and $\vec{w}$. Then if (x, y, z) are the coordinates of the point X they mush verify the following equation:

$$0 = \begin{vmatrix} x-a & y-b & z-c \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

6.1 Tangent line and normal plane

Definition The *tangent line* to a curve C at the point P is the straight line that has the same tangent vector at P as the curve C. We say that the tangent line has order of contact 1 with the curve C at the point P.

Thus, the tangent line to the curve C with tangent vector $\vec{t} = (v_1, v_2, v_3)$ at the point $P = (a, b, c)$ has the following parametrization: $\vec{r}(\lambda) = \overrightarrow{OP} + \lambda\vec{t}$, this is,

$$\begin{cases} x(\lambda) = a + \lambda v_1 \\ y(\lambda) = b + \lambda v_2, \quad \text{with } \lambda \in \mathbb{R}. \\ z(\lambda) = c + \lambda v_3, \end{cases}$$

Definition The *normal plane* to a curve C at a point P is the orthogonal plane to the tangent line that contains the point P.

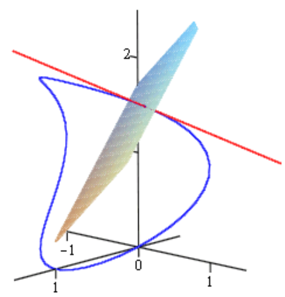
Let Π be the normal plane of the curve C with tangent vector $\vec{t} = (t_1, t_2, t_3)$ at the point $P = (a, b, c)$. A point X belongs to the normal plane if it verifies the following vectorial equation:

$$\overrightarrow{PX} \cdot \vec{t} = 0$$

That is, if (x, y, z) are the coordinates of the point X, the implicit equations of the plane is:

$$(x - a)t_1 + (y - b)t_2 + (z - c)t_3 = 0.$$

The following plot represents the tangent line and the normal plane of the curve with parametrization $\vec{r}(t) = (1 - \sin^2(t), \sin(t), 1 + \cos(t))$, $t \in [0, 2\pi)$, at the point $P = \vec{r}(\pi/4)$.



6.2 Principal normal line and osculating plane. Osculating circle.

Definition. The *principal normal line* to a curve C at a point P is the straight line that passes through the point P and whose direction vector is the normal vector to the curve at P. Thus the normal line to the curve C with normal vector $\vec{n} = (n_1, n_2, n_3)$ at the point $P = (a, b, c)$ has the following natural parametrization: $\vec{r}(\lambda) = \overrightarrow{OP} + \lambda\vec{n}$, this is,

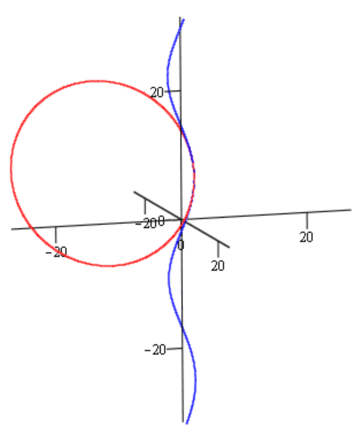
$$\begin{cases} x(\lambda) = a + \lambda n_1 \\ y(\lambda) = b + \lambda n_2, \\ z(\lambda) = c + \lambda n_3, \end{cases} \quad \text{with } \lambda \in \mathbb{R}.$$

Definition. The *osculating plane* to the curve C at a point P is the plane that contains the tangent and normal lines to the curve at the point P.

Let C be the curve with tangent vector $\vec{t} = (t_1, t_2, t_3)$ and normal vector $\vec{n} = (n_1, n_2, n_3)$ at the point $P = (a, b, c)$. A point X belongs to the osculating plane if and only if $\overrightarrow{PX}$ is a linear combination of the vectors $\vec{t}$ and $\vec{n}$. Then the coordinates (x, y, z) of X verify the following equation:

$$0 = \begin{vmatrix} x-a & y-b & z-c \\ t_1 & t_2 & t_3 \\ n_1 & n_2 & n_3 \end{vmatrix}.$$

Definition. We call *osculating circle* of the curve C at a point $P \in C$ the circle contained in the osculating plane of the curve C at P whose center, called *center of the curvature*, lies on the normal line and whose radius is $R(s_0) = 1/\kappa(s_0)$. See the picture bellow.



The osculating circle has *order of contact two* with the curve at a point $P \in C$; this means that it has the same tangent vector and the same curvature than the curve at the point P.

The center Z of the osculating circle at the point $P = \vec{\alpha}(s_0)$ satisfies the following equation:

$$\overrightarrow{OZ} = \overrightarrow{OP} + R(s_0)\vec{n}(s_0),$$

and the equations of the osculating circle is:

$$||\overrightarrow{ZX}|| = R(s_0).$$

6.2.1 Example.

The *evolute* of a curve is the locus of centers of curvature. Let us compute the evolute of the parabola contained in the plane $z = 0$, of equation $y = \frac{1}{2}x^2$. A parametrization of the parabola is $\vec{r}(t) = (t, \frac{1}{2}t^2, 0)$. By differentiating $\vec{r}(t)$ we obtain:

$$\vec{r}\,'(t) = (1, t, 0)\,, \quad \vec{r}\,''(t) = (0, 1, 0)\,, \quad \vec{r}\,'(t) \wedge \vec{r}\,''(t) = (0, 0, 1),$$

thus,

$$\vec{t}(t) = \tfrac{1}{\sqrt{1+t^2}}\,(1, t, 0)\,, \quad \vec{b}(t) = (0, 0, 1), \quad \vec{n}(t) = \tfrac{1}{\sqrt{1+t^2}}\,(-t, 1, 0)\,.$$

and the radius of curvature is given by the function:

$$R(t) = \tfrac{\|\vec{r}\,'(t)\|^3}{\|\vec{r}\,'(t)\wedge\vec{r}\,''(t)\|} = \left(1 + t^2\right)^{3/2}.$$

Thus, the locus of centers of curvature of the curve has the following parametrization:

$$\begin{aligned}
\vec{\beta}(t) &= \vec{r}(t) + R(t)\vec{n}(t) \\
&= \left(t, \tfrac{1}{2}t^2, 0\right) + \left(1 + t^2\right)^{3/2} \tfrac{1}{\sqrt{1+t^2}}\,(-t, 1, 0) \\
&= \left(t, \tfrac{1}{2}t^2, 0\right) + (1 + t^2)\,(-t, 1, 0) \\
&= \left(-t^3, 1 + \tfrac{3}{2}t^2, 0\right).
\end{aligned}$$

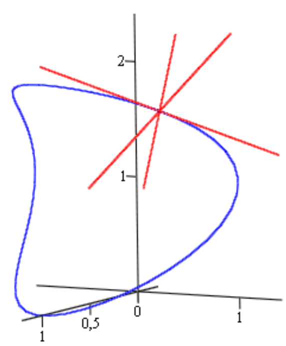
Note that the components of $\vec{\beta}(t)$ satisfy $y = 1 - \frac{3}{2}x^{2/3}$ and $z = 0$.

6.3 Binormal line and rectifying plane. Torsion.

Definition. The *binormal line* to the curve C at the point P is the line containing the point P and whose direction vector is the binormal vector to C at P. Thus the binormal line to the curve C with binormal vector $\vec{b} = (b_1, b_2, b_3)$ at the point $P = (a, b, c)$ has the following natural parametrization: $\vec{r}(\lambda) = \overrightarrow{OP} + \lambda\vec{b}$, this is,

$$\begin{cases} x(\lambda) = a + \lambda b_1 \\ y(\lambda) = b + \lambda b_2, \\ z(\lambda) = c + \lambda b_3, \end{cases} \quad \text{with } \lambda \in \mathbb{R}.$$

The following picture represents the tangent, normal and binormal lines of the curve with parametrization $\vec{r}(t) = (1 - \sin^2(t), \sin(t), 1 + \cos(t))$, $t \in [0, 2\pi)$, at the point $P = \vec{r}(\pi/4)$:



Definition. The *rectifying plane* of the curve C at the point $P = (a, b, c)$ is the plane containing the tangent and binormal lines to the curve at the point P. Thus the rectifying plane has the following vectorial equation:

$$\overrightarrow{PX} \cdot \vec{n} = 0 \Longleftrightarrow (x - a)\, n_1 + (y - b)\, n_2 + (z - c)\, n_3 = 0.$$

Remark. If $\tau(s) = 0$, $\forall s$, then $\vec{b}'(s) = \vec{0}$, the binormal vector is constant and the curve C is contained in the osculating plane. Therefore, C is a plane curve.

Proposition. Let C be a curve of class C^3, then

$$C \text{ is plane if and only if the torsion is zero.}$$

6.4 Examples

1. Let C be a curve with parametrization $\vec{r}(t) = (a\cos(t),\ a\sin(t),\ b)$, $t \in [0, 2\pi)$ and $a \neq 0, 1$. Let us compute the tangent, normal and binormal lines and de tangent, osculating and rectifying planes at a generic point $X_0 \in C$ and at the point $P = (0, a, b)$.

   (a) By differentiating $\vec{r}(t)$ we obtain

   $$\begin{aligned} \vec{r}\,'(t) &= (-a\sin(t),\ a\cos(t),\ 0), \\ \vec{r}\,''(t) &= (-a\cos(t),\ -a\sin(t),\ 0) \end{aligned}$$

   and therefore $||\vec{r}\,'(t)|| = a$ and

   $$\begin{aligned} \vec{t}(t) &= \tfrac{\vec{r}\,'(t)}{||\vec{r}\,'(t)||} = (-\sin(t),\ \cos(t),\ 0), \\ \vec{r}\,'(t) \wedge \vec{r}\,''(t) &= \begin{vmatrix} i & j & k \\ -a\sin(t) & a\cos(t) & 0 \\ -a\cos(t) & -a\sin(t) & 0 \end{vmatrix} = (0, 0, a^2) \\ \vec{b}(t) &= \tfrac{\vec{r}\,'(t)\wedge\vec{r}\,''(t)}{||\vec{r}\,'(t)\wedge\vec{r}\,''(t)||} = (0, 0, 1), \\ \vec{n}(t) &= \vec{b}(t) \wedge \vec{t}(t) = \begin{vmatrix} i & j & k \\ -\sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos(t), \sin(t)). \end{aligned}$$

   Remark. Note that the Frenet frame $\left(\vec{t}(t), \vec{n}(t), \vec{b}(t)\right)$ is a positive-oriented frame as

   $$\begin{vmatrix} -\sin(t) & -\sin(t) & 0 \\ \cos(t) & \sin(t) & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 > 0.$$

   Remark. Note that as $\vec{b}(t)$ is constant vector the curve is plane and it is contained at the osculating plane. A point X is at the osculating plane at a generic point $X_0 = \vec{r}(t_0) = (a\cos(t_0),\ a\sin(t_0),\ b)$ if and only if $\overrightarrow{X_0X} \cdot \vec{b}(t) = 0$. Hence the equation of the osculating plane at X_0 is:

   $$(x - a\cos(t_0),\ y - a\sin(t_0),\ z - b) \cdot (0, 0, 1) = 0 \Longleftrightarrow z = b.$$

At $P = (0, a, b) = \vec{r}(\frac{\pi}{2})$ we have:

$$\begin{aligned}
\vec{t}(\tfrac{\pi}{2}) &= (-\sin(\tfrac{\pi}{2}),\ \cos(\tfrac{\pi}{2}),\ 0) = (-1, 0, 0), \\
\vec{n}(\tfrac{\pi}{2}) &= (\cos(\tfrac{\pi}{2}), \sin(\tfrac{\pi}{2}), 0) = (0, 1, 0), \\
\vec{b}(\tfrac{\pi}{2}) &= (0, 0, 1).
\end{aligned}$$

(b) The tangent line at an arbitrary point $X_0 = \vec{r}(t_0)$ is the following:

$$\begin{aligned}
\vec{r}_t(\lambda) &= \vec{r}(t_0) + \lambda\vec{t}(t_0) \\
&= (a\cos(t_0),\ a\sin(t_0),\ b) + \lambda\,(-\sin(t_0),\ \cos(t_0),\ 0) \\
&= (a\cos(t_0) - \lambda\sin(t_0),\ a\sin(t_0) + \lambda\cos(t_0),\ b).
\end{aligned}$$

The principal normal line at an arbitrary point $X_0 = \vec{r}(t_0)$ is the following:

$$\begin{aligned}
\vec{r}_n(\lambda) &= \vec{r}(t_0) + \lambda\vec{n}(t_0) \\
&= (a\cos(t_0),\ a\sin(t_0),\ b) + \lambda\,(\cos(t_0),\ \sin(t_0),\ 0) \\
&= (a\cos(t_0) + \lambda\cos(t_0),\ a\sin(t_0) + \lambda\sin(t_0),\ b) \\
&= ((a+\lambda)\cos(t_0),\ (a+\lambda)\sin(t_0),\ b).
\end{aligned}$$

The binormal line at an arbitrary point $X_0 = \vec{r}(t_0)$ is the following:

$$\begin{aligned}
\vec{r}_b(\lambda) &= \vec{r}(t_0) + \lambda\vec{b}(t_0) \\
&= (a\cos(t_0),\ a\sin(t_0),\ b) + \lambda(0, 0, 1) \\
&= (a\cos(t_0) - \lambda\sin(t_0),\ a\sin(t_0) + \lambda\cos(t_0),\ b).
\end{aligned}$$

The tangent, normal and binormal lines at $P = \vec{r}(\frac{\pi}{2})$ are given by the following parametric equations:

$$\begin{aligned}
\vec{r}_t(\lambda) &= (a\cos(\tfrac{\pi}{2}) - \lambda\sin(\tfrac{\pi}{2}),\ a\sin(\tfrac{\pi}{2}) + \lambda\cos(\tfrac{\pi}{2}),\ b) = (-\lambda,\ a,\ b), \\
\vec{r}_n(\lambda) &= ((a+\lambda)\cos(\tfrac{\pi}{2}),\ (a+\lambda)\sin(\tfrac{\pi}{2}),\ b) = (0,\ a+\lambda,\ b), \\
\vec{r}_b(\lambda) &= (a\cos(\tfrac{\pi}{2}) - \lambda\sin(\tfrac{\pi}{2}),\ a\sin(\tfrac{\pi}{2}) + \lambda\cos(\tfrac{\pi}{2}),\ b) = (-\lambda,\ a,\ b).
\end{aligned}$$

(c) A point X is at the normal plane at $X_0 = \vec{r}(t_0) = (a\cos(t_0),\ a\sin(t_0),\ b)$ if and only if $\overrightarrow{X_0X} \cdot \vec{t}(t_0) = 0$. Hence, the equation of the normal plane is:

$$(x - a\cos(t_0),\ y - a\sin(t_0),\ z - b) \cdot (-\sin(t_0),\ \cos(t_0),\ 0) = 0$$

that is,

$$-(x - a\cos(t_0))\sin(t_0) + (y - a\sin(t_0))\cos(t_0) = 0.$$

A point X is at the rectifying plane at $X_0 = \vec{r}(t_0)$ if and only if $\overrightarrow{X_0X} \cdot \vec{n}(t_0) = 0$. Hence the equation of the rectifying plane is:

$$(x - a\cos(t_0),\ y - a\sin(t_0),\ z - b) \cdot (\cos(t_0),\ \sin(t_0),\ 0) = 0$$

that is,

$$(x - a\cos(t_0))\cos(t_0) + (y - a\sin(t_0))\sin(t_0) = 0.$$

And the equation of the osculating plane at X_0 is $z = b$.
The normal, rectifying and osculating planes at $P = \vec{r}(\frac{\pi}{2})$ are given by the following implicit equations:

$$\begin{aligned} \Pi_n &\equiv -(x - a) = 0, \\ \Pi_r &\equiv y = 0, \\ \Pi_o &\equiv z = b. \end{aligned}$$

2. *Viviani's curve.* Let C be the curve with parametrization:

$$\vec{r}(t) = \left(1 + \cos t,\ \sin t,\ 2\sin\tfrac{t}{2}\right), \quad t \in [0, 2\pi).$$

Let us compute the tangent, normal and binormal lines and de tangent, osculating and rectifying planes at the point $P = (0, 0, 2)$.

(a) By differentiating $\vec{r}(t)$ we obtain

$$\begin{aligned} \vec{r}\,'(t) &= (-\sin(t),\ \cos(t),\ \cos(\tfrac{t}{2})), \\ \vec{r}\,''(t) &= (-\cos(t),\ -\sin(t),\ -\tfrac{1}{2}\sin(\tfrac{t}{2})) \end{aligned}$$

and therefore $||\vec{r}\,'(t)|| = \sqrt{1 + \cos^2(\frac{t}{2})}$. We have: $P = \vec{r}(\pi)$, as $1 + \cos t = 0$, $\sin t = 0$ and $2\sin\frac{t}{2} = 2$ if and only if $t = \pi$. We have

$$\begin{aligned} \vec{r}\,'(\pi) &= (0,\ -1,\ 0), \\ \vec{r}\,''(\pi) &= (1,\ 0,\ -\tfrac{1}{2}), \end{aligned}$$

and therefore

$$
\begin{aligned}
\vec{t}(\pi) &= \frac{\vec{r}\,'(\pi)}{||\vec{r}\,'(\pi)||} = (0,\ -1,\ 0),\\
\vec{r}\,'(\pi)\wedge\vec{r}\,''(\pi) &= \begin{vmatrix} i & j & k\\ 0 & -1 & 0\\ 1 & 0 & -\frac{1}{2}\end{vmatrix} = (\tfrac{1}{2}, 0, 1),\\
\vec{b}(\pi) &= \frac{\vec{r}\,'(\pi)\wedge\vec{r}\,''(\pi)}{||\vec{r}\,'(\pi)\wedge\vec{r}\,''(\pi)||} = \frac{1}{\sqrt{\frac{1}{4}+1}}(\tfrac{1}{2}, 0, 1) = \left(\tfrac{1}{\sqrt{5}}, 0, \tfrac{2}{\sqrt{5}}\right),\\
\vec{n}(\pi) &= \vec{b}(\pi)\wedge\vec{t}(\pi) = \begin{vmatrix} i & j & k\\ 0 & -1 & 0\\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}}\end{vmatrix} = \left(-\tfrac{2}{\sqrt{5}}, 0, \tfrac{1}{\sqrt{5}}\right).
\end{aligned}
$$

The tangent line at $P = \vec{r}(\pi)$ is the following:

$$
\begin{aligned}
\vec{r}_t(\lambda) &= \vec{r}(\pi) + \lambda\vec{t}(\pi)\\
&= (0,0,2) + \lambda(0,-1,0)\\
&= (0,-\lambda,2).
\end{aligned}
$$

The principal normal line at $P = \vec{r}(\pi)$ is the following:

$$
\begin{aligned}
\vec{r}_n(\lambda) &= \vec{r}(\pi) + \lambda\vec{n}(\pi)\\
&= (0,0,2) + \lambda\left(-\tfrac{2}{\sqrt{5}}, 0, \tfrac{1}{\sqrt{5}}\right)\\
&= \left(-\tfrac{2}{\sqrt{5}}\lambda, 0, 2 + \tfrac{1}{\sqrt{5}}\lambda\right).
\end{aligned}
$$

The binormal line at $P = \vec{r}(\pi)$ is the following:

$$
\begin{aligned}
\vec{r}_b(\lambda) &= \vec{r}(\pi) + \lambda\vec{b}(\pi)\\
&= (0,0,2) + \lambda\left(\tfrac{1}{\sqrt{5}}, 0, \tfrac{2}{\sqrt{5}}\right)\\
&= \left(\lambda\tfrac{1}{\sqrt{5}}, 0, 2 + \lambda\tfrac{2}{\sqrt{5}}\right).
\end{aligned}
$$

(b) A point X is at the normal plane at $P = \vec{r}(\pi)$ if and only if $\overrightarrow{PX}\cdot\vec{t}(\pi) = 0$. Hence, the equation of the normal plane is:

$$(x,\ y,\ z-2)\cdot(0,-1,0) = 0$$

that is, $\Pi_n \equiv y = 0$.

A point X is at the rectifying plane at $P = \vec{r}(\pi)$ if and only if $\overrightarrow{PX} \cdot \vec{n}(\pi) = 0$. Hence the equation of the rectifying plane is:

$$(x,\ y,\ z-2) \cdot \left(-\tfrac{2}{\sqrt{5}}, 0, \tfrac{1}{\sqrt{5}}\right) = 0$$

that is,

$$\Pi_r \equiv -\tfrac{2}{\sqrt{5}}x + (z-2)\,\tfrac{1}{\sqrt{5}} = 0 \Longleftrightarrow -2x + z = 2.$$

A point X is at the osculating plane at $P = \vec{r}(\pi)$ if and only if $\overrightarrow{PX} \cdot \vec{b}(\pi) = 0$. Hence the equation of the osculating plane is:

$$(x,\ y,\ z-2) \cdot \left(\tfrac{1}{\sqrt{5}}, 0, \tfrac{2}{\sqrt{5}}\right) = 0$$

that is,

$$\Pi_o \equiv \tfrac{1}{\sqrt{5}}x + (z-2)\,\tfrac{2}{\sqrt{5}} = 0 \Longleftrightarrow x + 2z = 4.$$

7 Notables curves

7.1 Some planar curves

7.1.1 Cycloids

The *cycloid* is the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line without slippage. Find the singular point of the cycloid with parametrization:

$$\vec{r}(\lambda) = (\lambda - \sin\lambda,\ 1 - \cos\lambda), \quad \lambda \in \mathbb{R}.$$

$$\vec{r}(t) = (a\lambda - a\sin\lambda,\ a - a\cos\lambda), \quad \lambda \in \mathbb{R}.$$

$$\vec{r}(t) = (a\lambda - b\sin\lambda,\ a - b\cos\lambda), \quad \lambda \in \mathbb{R}.$$

Prolate cycloid $a < b$
Curtate cycloid $a > b$
Epicicloide
Hipocicloide

7.1.2 Espirales

7.1.3 The Catenary

7.2 Helixes or curves of constant slope

Definition. The *helixes* or *curves of constant slope* are defined by the property that the tangent forms a constant angle with a fixed vector $\vec{u} \in \mathbb{R}^3$.

Proposition. The constancy of $\frac{\tau(s)}{\kappa(s)}$ where $\tau(s)$ is the torsion function and $\kappa(s)$ is the curvature function for a curve is equivalent to the constancy of the angle between the curve and a fixed vector.

Lancret theorem (1802). A curve with regular parametric representation $\vec{r}(t)$, $t \in I \subseteq \mathbb{R}$, is an helix if and only the ratio $\frac{\tau(s)}{\kappa(s)}$ is constant $\forall t \in I$.

Proof. Let $\vec{\alpha}\colon J \subseteq \mathbb{R} \longrightarrow \mathbb{R}^3$ be a natural parametric representation of an helix whose tangent forms a constant angle θ with a vector $\vec{u} \in \mathbb{R}^3$, $||\vec{u}|| = 1$; that is,

$$\vec{t}(s) \cdot \vec{u} = \cos\theta.$$

By taking the derivative in the previous equation we obtain

$$\vec{t}\,'(s) \cdot \vec{u} + \vec{t}(s) \cdot \vec{u}' = \vec{n}(s) \cdot \vec{u} = 0.$$

Thus, the vector $\vec{u}$ is orthogonal to $\vec{n}(s)$ and can be written as a combination of the vectors $\vec{t}(s)$ and $\vec{b}(s)$; that is,

$$\vec{u} = a_1\vec{t}(s) + a_2\vec{b}(s), \text{ with } a_1^2 + a_2^2 = 1.$$

Let assume $a_1 = \cos\alpha$ and $a_2 = \sin\alpha$. By taking the derivative of the equation $\vec{n}(s) \cdot \vec{u} = 0$ and by using the second Frenet-Serret formulas ,we obtain

$$\begin{aligned} 0 &= \vec{n}'(s) \cdot \vec{u} \\ &= (-\kappa(s)\vec{t}(s) + \tau(s)\vec{b}(s)) \cdot (\vec{t}(s)\cos\alpha + \vec{b}(s)\sin\alpha) \\ &= -\kappa(s)\cos\alpha + \tau(s)\sin\alpha. \end{aligned}$$

Therefore,

$$\tfrac{\kappa(s)}{\tau(s)} = \frac{\sin\alpha}{\cos\alpha} = \tan\alpha \in \mathbb{R}.$$

Reciprocally, if a regular curve satisfies the condition:

$$\tfrac{\kappa(s)}{\tau(s)} = a \in \mathbb{R},$$

then we can find an angle θ such that: $a = \tan\theta$. Taking into account the Frenet-Serret formulas we have:

$$\begin{aligned} \vec{t}\,'(s) &= a\tau(s)\vec{n}(s) = \tan\theta\tau(s)\vec{n}(s), \\ \vec{b}'(s) &= -\tau(s)\vec{n}(s). \end{aligned}$$

Therefore by differentiating the vector $\vec{u}(s) = \vec{t}(s)\cos\theta + \vec{b}(s)\sin\theta$ and substituting the previous formulas we obtain

$$\begin{aligned} \vec{u}'(s) &= (\vec{t}(s)\cos\theta + \vec{b}(s)\sin\theta)' \\ &= \vec{t}\,'(s)\cos\theta + \vec{b}'(s)\sin\theta \\ &= \tau(s)\,(\tan\theta\cos\theta - \sin\theta)\,\vec{n}(s) \\ &= 0. \end{aligned}$$

Then the vector $\vec{u}(s)$ is constant and satisfies $\vec{u} \cdot \vec{t}(s) = \cos\theta$. Therefore the curve is an helix.

7.2.1 Example

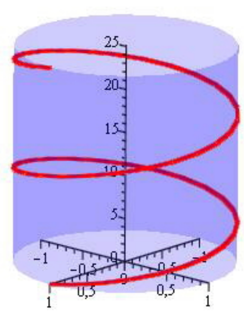
Find that the curve with parametrization

$$\vec{r}(t) = (t\cos t,\ t\sin t, t)\,, \quad t \in [0, 2\pi),$$

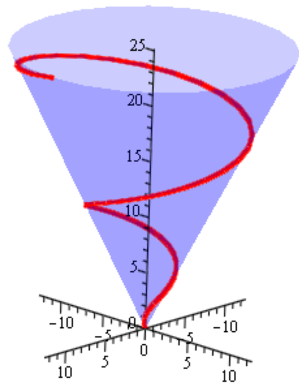
is contained in a cone. Find the curvature and the torsion at the point that corresponds to vertex of the cone.

7.2.2 Circular helix

The circular helixes are a special case of a class or curves called *helixes*.



7.2.3 Conic helix



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